

STRUCTURE AND K-THEORY OF CROSSED PRODUCTS BY PROPER ACTIONS

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ABSTRACT. We study the C^* -algebra crossed product $C_0(X) \rtimes G$ of a locally compact group G acting properly on a locally compact Hausdorff space X . Under some mild extra conditions, which are automatic if G is discrete or a Lie group, we describe in detail, and in terms of the action, the primitive ideal space of such crossed products as a topological space, in particular with respect to its fibering over the quotient space $G \backslash X$. We also give some results on the K-theory of such C^* -algebras. These more or less compute the K-theory in the case of isolated orbits with non-trivial (finite) stabilizers. We also give a purely K-theoretic proof of a result due to Paul Baum and Alain Connes on K-theory with complex coefficients of crossed products by finite groups.

0. INTRODUCTION

The C^* -algebra crossed products $C_0(X) \rtimes G$ associated to finite group actions on smooth compact manifolds give the simplest non-trivial examples of noncommutative spaces. Such group actions also play a role in various other parts of mathematics and physics, *e.g.* the linear action of a Weyl group on a complex torus in representation theory.

As soon as the action of G on X is not free, the primitive ideal space of the crossed product $C_0(X) \rtimes G$ is non-Hausdorff, although the quotient space $G \backslash X$ is Hausdorff. In fact $\text{Prim}(C_0(X) \rtimes G)$ fibres over $G \backslash X$, in a canonical way, with finite fibres. As a fibration of sets this is easy enough to describe in direct, geometric terms, and it is ‘well-known to experts’: the primitive ideal space is in a natural set bijection with $G \backslash \text{Stab}(X)^\wedge$, where $\text{Stab}(X)^\wedge = \{(x, \pi) \mid x \in X, \pi \in \widehat{G_x}\}$, and the first coordinate projection defines the required fibration; the fibre over Gx is thus the unitary dual $\widehat{G_x}$ of the stabilizer G_x . However, for purposes, for example, of K-theory computation, what is wanted here is a description of the open sets of $G \backslash \text{Stab}(X)^\wedge$ which correspond to the open subsets of the primitive ideal space equipped with the Fell topology, since this describes the space of ideals of $C_0(X) \rtimes G$ and potentially leads to a method of K-theory computation using excision. This description is the main contribution of this article.

Proper actions of general locally compact groups naturally generalize actions of compact, or finite, groups – because every proper action is ‘locally induced’ from actions by compact subgroups. Such crossed products are important in operator algebras, because of the Baum-Connes conjecture. For an amenable, locally compact group G with G -compact universal proper G -space \underline{EG} , Kasparov’s Fredholm representation ring $R(G) := \text{KK}^G(\mathbb{C}, \mathbb{C})$ is canonically isomorphic to the K-theory of $C_0(\underline{EG}) \rtimes G$. This is one statement of the Baum-Connes conjecture (and is due to Higson and Kasparov). In fact there are several possible statements of the Baum-Connes conjecture, but the general idea is that computation of K-theory of the C^* -algebra crossed products involving *arbitrary* group actions can be, in certain

The research for this paper was partially supported by the German Research Foundation (SFB 478 and SFB 878) and the EU-Network Quantum Spaces Noncommutative Geometry (Contract No. HPRN-CT-2002-00280).

circumstances, be reduced to the case where the action is proper. Thus the importance of computing K-theory for proper actions. Since the analysis of the structure of the crossed product runs along similar lines in the case of proper actions of locally compact groups as for actions of compact (or finite) groups, we treat the more general problem in this article.

If G is a locally compact group acting properly on X , we show that $C_0(X) \rtimes G$ is isomorphic to a certain generalized fixed-point algebra, denoted $C_0(X \times_G \mathcal{K})$, with respect to the diagonal action of G on $C_0(X) \otimes \mathcal{K}(L^2(G))$, with action of G on the second factor given by the adjoint of the right regular representation ρ of G (in fact, we show a more general result along these lines for crossed products $B \rtimes G$ where B is a ‘fibred’ over some proper G -space X). It follows that $C_0(X) \rtimes G$ is the algebra of C_0 -sections of a continuous bundle of C^* -algebras over the orbit space $G \backslash X$ with fiber over Gx isomorphic to the fixed point algebra $\mathcal{K}(L^2(G))^{G_x}$, where G_x denotes the (compact) stabilizer at x which acts via conjugation by the restriction of right regular representation of G to G_x .

This result has been shown by Bruce Evans in [25] for compact group actions, but we are not aware of any reference for the more general class of proper actions.

The Peter-Weyl theorem implies that the fixed-point algebras $\mathcal{K}(L^2(G))^{G_x}$ decompose into direct sums of algebras of compact operators indexed over the unitary duals \widehat{G}_x of G_x . It follows that the primitive ideal space of $A = C_0(X) \rtimes G$ is in a natural set bijection with $G \backslash \text{Stab}(X)^\wedge$, where $\text{Stab}(X)^\wedge = \{(x, \pi) \mid x \in X, \pi \in \widehat{G}_x\}$, which is a bundle, by the first coordinate projection, over the space $G \backslash X$ of orbits, the fibre over Gx being the unitary dual \widehat{G}_x of the stabilizer G_x (*c.f.* the first paragraph of this Introduction).

It is possible to describe the topology on $G \backslash \text{Stab}(X)^\wedge$ corresponding to the Fell topology on $\text{Prim}(A)$, in direct terms of the action. We do this in the case where the action of G on X satisfies Palais’s slice property, which means that X is locally induced from the *stabilizer* subgroups of G (see §1 for this notion). By a famous theorem of Palais, this property is always satisfied if G is a Lie group.

In general, $G \backslash X$ is always an open subset of $G \backslash \text{Stab}(X)^\wedge$, and therefore corresponds to an ideal of $C_0(X) \rtimes G$, as remarked above. Therefore to compute the K-theory of the crossed product, it suffices to compute the K-theory of the quotient space $G \backslash X$ together with the boundary maps in the associated six-term sequence. In the case of isolated fixed points and discrete G this is fairly straightforward, at least up to torsion; in general, it is non-trivial, but we show in examples how the knowledge of the ideal structure of $C_0(X) \rtimes G$ can still help in K-theory computations, even when fixed points are not isolated.

The problem of computing the K-theory *in general* does not have an obvious solution. Indeed, it may not have any solution at all. However, if one ignores torsion, the problem gets much easier, at least for compact group actions. The reason is that if G is compact, the K-theory of $C_0(X) \rtimes G$ is a module over the representation ring $\text{Rep}(G)$, and similarly the K-theory tensored by \mathbb{C} is a module over $\text{Rep}(G) \otimes_{\mathbb{Z}} \mathbb{C}$. For many groups G of interest, like finite groups, or connected groups, the complex representation ring $\text{Rep}(G) \otimes_{\mathbb{Z}} \mathbb{C}$ is quite a tractable ring, and the module structure of the complex K-theory of $C_0(X) \rtimes G$ gives significant additional information when used in conjunction with a ‘localization principal’ developed mainly by Atiyah and Segal in the 1960’s in connection with the Index Theorem. These ideas were exploited by Paul Baum and Alain Connes in the 1980’s to give a very beautiful formula for the K-theory (tensored by \mathbb{C}) of the crossed product in the case of *finite* group actions. We give a full proof of the theorem of Baum and Connes in this article, without attempting to generalize it to proper actions of locally compact groups. The formula of Baum and Connes means that the difficulty in computing

$K_*(C_0(X) \rtimes G)$ for finite group actions, is concentrated in the problem of computing the torsion subgroup. We do not shed much light on this problem.

This paper is to some extent expository. Our goal is to provide a readable synopsis of what is known, and what can be proved without too much difficulty, about crossed products by proper actions, and their K-theory. The paper is organized as follows: after giving some preliminaries on proper actions in §1 we give a detailed discussion of the bundle structure of $C_0(X) \rtimes G$ in §2. In §3 we discuss the natural bijection between $\text{Prim}(C_0(X) \rtimes G)$ and the quotient space $G \backslash \text{Stab}(X)^\wedge$ and in §4 we show that this bijection is a homeomorphism for a quite naturally defined topology on $\text{Stab}(X)^\wedge$ if the action satisfies the slice property. All K-theoretic discussions can be found in §5.

All spaces considered in this paper with the obvious exceptions of primitive ideal spaces of C*-algebras and the like, are assumed locally compact Hausdorff.

A good part of this paper has been written while the first named author visited the University of Victoria in Summer 2008. He is very grateful to the second named author and his colleagues for their warm hospitality during that stay! The authors are also grateful for some useful conversations with Wolfgang Lück and Jan Spakula.

1. PRELIMINARIES ON PROPER ACTIONS

Assume that G is a locally compact group. Suppose that G acts on the (locally compact Hausdorff) space X . The action is *proper* if the map

$$G \times X \rightarrow X \times X : (g, x) \mapsto (gx, x)$$

is proper – i.e. if inverse images of compact sets are compact. Since X is locally compact, this is equivalent to the condition that for every compact subset $K \subseteq X$ the set $\{g \in G : g^{-1}K \cap K \neq \emptyset\}$ is compact in G . In particular, *any* action of a compact group on a locally compact Hausdorff space is proper. We use Palais’s fundamental paper [40] as a basic reference.

It is immediately clear that if G acts properly on X then the stabilizers $G_x := \{g \in G : gx = x\}$ are compact. Moreover, as one can show (e.g. see [40, Theorem 1.2.9]) without much difficulty, the quotient space $G \backslash X$ endowed with the quotient topology is a locally compact Hausdorff space.

If $H \subseteq G$ is a closed subgroup of G which acts on some space Y , then the induced G -space $G \times_H Y$ is defined as the quotient space $(G \times Y)/H$ with respect to the diagonal action $h \cdot (g, y) = (gh^{-1}, hy)$. The action of H on $G \times Y$ is obviously free; it is a good exercise to prove that it is also proper (see [40, §1.3]). Hence $G \times_H Y$ is a locally compact Hausdorff space. It carries a natural G -action by left translation on the first factor. The process of going from H acting on Y to G acting on $G \times_H Y$ is called ‘induction.’

Proposition 1.1 (c.f. [14, Corollary]). *Suppose that X is a G -space and H is a closed subgroup of G . Then the following are equivalent:*

- (1) *There exists an H -space Y such that $X \cong G \times_H Y$ as G -spaces.*
- (2) *There exists a continuous G -map $\varphi : X \rightarrow G/H$.*

In case of (1), the corresponding G -map $\varphi : G \times_H Y \rightarrow G/H$ is given by $\varphi([g, y]) = gH$ and in case of (2), the corresponding H -space Y is the closed subset $Y := \varphi^{-1}(\{eH\})$ of X . The G -homeomorphism from $\Phi : G \times_H Y \rightarrow X$ is then given by $\Phi([g, y]) = gy$.

Following Palais [40], we shall call a closed subset $Y \subseteq X$ a *global H -slice* if there exists a map $\varphi : X \rightarrow G/H$ as in part (2) of the above theorem with $Y = \varphi^{-1}(\{eH\})$, i.e., $Y \subseteq X$ is a global H -slice if and only if Y is H -invariant and $X \cong G \times_H Y$ as a G -space. If U is a G -invariant open subset of X , then we say

that $Y \subset U$ is a *local H -slice* of X , if Y is a global H -slice for the G -space U . The following observation is well known, but by lack of a direct reference, we give the proof.

Lemma 1.2. *Suppose that H is a closed subgroup of G and that Y is an H -space. Then $G \times_H Y$ is a proper G -space if and only if Y is a proper H -space.*

Proof. Suppose first that $X = G \times_H Y$ is a proper G -space. Then it is also a proper H -space, and since $\varphi : Y \rightarrow G \times_H Y; \varphi(y) = [e, y]$ includes Y as an H -invariant closed subset of X , it must be a proper H -space, too.

Conversely, if Y is a proper H -space and $K \subseteq G \times_H Y$ is any compact set, then we may choose compact sets $C \subseteq G$ and $D \subseteq Y$ such that $K \subseteq C \times_H D := \{[c, d] \in G \times_H Y : c \in C, d \in D\}$. Let $F := \{h \in H : h^{-1}D \cap D \neq \emptyset\}$. Since H acts properly on Y , F is a compact subset of H . Suppose now that $g \in G$ such $g^{-1}(C \times_H D) \cap (C \times_H D) \neq \emptyset$. Then there exist $c_1, c_2 \in C$ and $d_1, d_2 \in D$ such that $[g^{-1}c_1, d_1] = [c_2, d_2]$, which in turn implies that there exists an element $h \in H$ such that $(g^{-1}c_1h, h^{-1}d_1) = (c_2, d_2)$. Then $h \in F$ and $g^{-1}c_1h = c_2$ implies that $g = c_1hc_2^{-1} \in CFC^{-1}$, which is compact in G . \square

The above lemma shows in particular that every G -space which is induced from some compact subgroup L of G must be proper. This observation has a partial converse, as the following theorem of Abels ([1]) shows:

Theorem 1.3 (Abels). *Suppose that X is a proper G -space. Then for each $x \in X$ there exist a G -invariant open neighborhood U_x of x , a compact subgroup L_x of G , and a continuous G -map $\varphi_x : U_x \rightarrow G/L_x$. Thus $Y_x := \varphi_x^{-1}(\{L_x\})$ is a local L_x -slice for X and $U_x \cong G \times_{L_x} Y_x$.*

Thus, combining the Theorem with Proposition 1.1, we see that every proper G -space is locally induced from compact subgroups.

Remark 1.4. We make two minor remarks about slices whose easy verification we leave to the reader:

- We may always choose L_x and φ_x in the theorem in such a way that $\varphi_x(x) = eL_x$, and hence $x \in Y_x$ (at the expense of changing the subgroup L_x to a conjugate subgroup).
- If $Y \subset U$ is a slice for the subgroup L_x then the isotropy subgroups of G in U are all subconjugate to L_x , i.e. are conjugate to subgroups of L_x .
- If $U \subseteq X$ is a G -invariant subset with L_x -slice Y_x then the intersection $Y_x \cap V$ is a L_x -slice for any given G -invariant subset $V \subseteq U$.

Example 1.5. Let $G := \prod_{n \in \mathbb{N}} \mathbb{Z}/2$, realize $\mathbb{Z}/2$ as $\{\pm 1\} \subset \mathbb{T}$, and let $X := \prod_{n \in \mathbb{N}} \mathbb{T}$, with action of G on X given by translation. This is a free and proper action of a compact, totally disconnected group. It is easy to construct many local slices. Let I be a small interval neighbourhood of $1 \in \mathbb{T}$, $J = I \cup -I$. A G -neighbourhood basis (in the sense of giving a neighbourhood in the quotient space) of $x := (1, 1, 1, \dots) \in X$ is supplied by basic product sets of the form $U = J \times \dots \times J \times \mathbb{T} \times \mathbb{T} \times \dots$. Thus any open G -invariant neighbourhood of x must contain one of these. Each of these open subsets has 2^k components, where k is the number of factors of J occurring, and under any G -map from U to a totally disconnected space like G itself must thus map the above open subset to a finite subset of the target. A slice at x produces a G -map from any sufficiently small one of these G -neighbourhoods, with target some G/L_x . Since this is also a totally disconnected space, the map must have image in a finite subset, since the subset must be G -invariant, it must be that G/L_x is itself finite. Hence all slices must use closed subgroups L_x of finite index.

In particular there is no slice through x with L_x the trivial subgroup of G , *i.e.* no slices through x with group L_x exactly equal to the isotropy group G_x .

If L_x is co-finite, it must be a subgroup of one of the ‘obvious’ ones $L_x = \{(x_i) \mid x_i = 1 \text{ if } i \leq n\}$, the quotient G/L_x G -equivariantly identifies with a finite product of $\mathbb{Z}/2$ ’s and it is easy to produce a slice, *i.e.* a G -map $U := J \times J \times \cdots J \times \mathbb{T} \times \mathbb{T} \times \cdots$ to G/L_x , by identifying $J \cong I \times \mathbb{Z}/2$ and using $Y_x := I \times \cdots \times I \times \mathbb{T} \times \mathbb{T} \times \cdots$.

In particular, whenever one has a slice, the stabilizer G_x of x is a closed subgroup of L_x , but $G_x \subseteq L_x$ may be strict. However, this can always be avoided when G is a *Lie group*. This is the content of the following well-known result of Palais.

Theorem 1.6 (Palais’s Slice Theorem). *Suppose that the Lie group G acts properly on the locally compact G -space X . Then for every $x \in X$ there exists an open G -invariant neighborhood U_x of x which admits a G_x -slice $Y_x \subseteq U_x$ with $x \in Y_x$.*

We should point out that Palais’s original theorem (see [40, Proposition 2.3.1]) is stated for completely regular proper spaces, and therefore is actually slightly more general; we will not need the extra generality here.

Motivated by Palais’s theorem we give the following

Definition 1.7. Let G be a locally compact group and let X be a proper G -space. We say that (G, X) satisfies *Palais’s slice property* (SP) if the conclusion of Theorem 1.6 holds for (G, X) , *i.e.*, if X is locally induced from the stabilizers G_x .

Remark 1.8. We emphasise that property (SP) for (G, X) implies that every point $x \in X$ has a G -invariant neighborhood U_x such that the stabilizers G_y for all $y \in U_x$ are sub-conjugate to G_x (*c.f.* Remark 1.4.)

2. PROPER ACTIONS AND C^* -ALGEBRA BUNDLES

If X is a locally compact G -space, then there is a corresponding action on the C^* -algebra $C_0(X)$ of all functions on X which vanish at ∞ given by $(g \cdot f)(g') = f(g^{-1}g')$. The main object of this paper is the study of the crossed product $C_0(X) \rtimes G$ in case where X is a proper G -space. For the general theory of crossed products we refer to Dana Williams’s book [48].

The construction of crossed products for group actions on spaces goes back to early work of Glimm (see [27]). Consider the space $C_c(G \times X)$ of continuous functions with compact supports on $G \times X$ equipped with convolution and involution given by the formulas

$$\varphi * \psi(g, x) = \int_G \varphi(t, x) \psi(t^{-1}g, t^{-1}x) dt \quad \text{and} \quad \varphi^*(g, x) = \Delta(g^{-1}) \overline{\varphi(g^{-1}, g^{-1}x)}.$$

Let $L^1(G, X)$ denote the completion of $C_c(G \times X)$ with respect to the norm $\|f\| = \int_G \|f(g, \cdot)\|_\infty dg$. Then $C_0(X) \rtimes G$ is the envelopping C^* -algebra of the Banach- $*$ -algebra $L^1(G, X)$. It enjoys the universal property for covariant representations of the pair $(C_0(X), G)$ as explained in detail in [48, Proposition 2.40].

It follows from [49] that the crossed product $C_0(X) \rtimes G$ for any proper G -space X has a canonical structure as an algebra of sections of a continuous C^* -algebra bundle over $G \backslash X$. In this section we want to give a more detailed description of this bundle.

Recall that if Z is any locally compact space, then a C^* -algebra A is called a $C_0(Z)$ -algebra (or an *upper semi-continuous bundle of C^* -algebras over Z*) if there exists a $*$ -homomorphism $\phi : C_0(Z) \rightarrow ZM(A)$, the center of the multiplier algebra of A , such that $\phi(C_0(Z))A = A$. For $z \in Z$ put $I_z := \phi(C_0(Z \setminus \{z\}))A$ and $A_z = A/I_z$. Then A_z is called the *fibre of A at z* . Then every $a \in A$ can be viewed as a section of the bundle of C^* -algebras $\{A_z : z \in Z\}$ via $a : z \mapsto a_z := a + I_z$.

The resulting positive function $z \mapsto \|a_z\|$ is always upper semi continuous and we have $\|a\| = \sup_{z \in Z} \|a_z\|$ for all $a \in A$. We say that A is a *continuous bundle of C^* -algebras over Z* if in addition all functions $z \mapsto \|a_z\|$ are continuous. We refer to [48, C.2] for the general properties of $C_0(Z)$ -algebras. In what follows we shall usually suppress the name of the structure map $\phi : C_0(Z) \rightarrow ZM(A)$ and we shall simply write fa for $\phi(f)a$ if $f \in C_0(Z)$ and $a \in A$.

A $*$ -homomorphism $\Psi : A \rightarrow B$ between two $C_0(Z)$ -algebras A and B is called $C_0(Z)$ -linear, if it commutes with the $C_0(Z)$ -actions, that is if $\Psi(fa) = f\Psi(a)$ for all $f \in C_0(Z)$, $a \in A$. A $C_0(Z)$ -linear homomorphism Ψ induces $*$ -homomorphisms $\Psi_z : A_z \rightarrow B_z$ for all $z \in Z$ by defining $\Psi_z(a + I_z^A) = \Psi(a) + I_z^B$ for all $z \in Z$, $a \in A$.

If A is a $C_0(Z)$ -algebra and G acts on A by $C_0(Z)$ -linear automorphisms, then the full crossed product $A \rtimes G$ is again a $C_0(Z)$ -algebra with respect to the composition

$$C_0(Z) \xrightarrow{\Phi} ZM(A) \xrightarrow{i_{M(A)}} ZM(A \rtimes G),$$

where $i_{M(A)} : M(A) \rightarrow M(A \rtimes G)$ denotes the extension to $M(A)$ of the canonical inclusion $i_A : A \rightarrow M(A \rtimes G)$ (see [48, Proposition 2.3.4] for the definition of i_A). The action α then induces actions α^z of G on each fiber A_z and it follows then from the exactness of the maximal crossed product functor that the fibre $(A \rtimes G)_z$ of the crossed product at a point $z \in Z$ coincides with $A_z \rtimes G$ (e.g., see [48, Theorem 8.4]). The following well-known lemma is often useful.

Lemma 2.1. *Suppose that $\Phi : A \rightarrow B$ is a $C_0(Z)$ -linear $*$ -homomorphism between the $C_0(Z)$ -algebras A and B . Then Φ is injective (resp. surjective, resp. bijective) if and only if every fibre map Φ_z is injective (resp. surjective, resp. bijective).*

Proof. Since $\|a\| = \sup_{z \in Z} \|a_z\|$ for all $a \in A$, it is clear that Φ is injective if Φ_z is injective for all $z \in Z$. Conversely assume Φ is injective. Then $B' := \Phi(A) \subseteq B$ is a $C_0(X)$ -subalgebra of B and there exists a $C_0(X)$ -linear inverse $\Phi^{-1} : B' \rightarrow A$ which induces fibre-wise inverses Φ_z^{-1} for $\Phi_z : A_z \rightarrow B'_z \subseteq B_z$.

Surjectivity of Φ clearly implies surjectivity of Φ_z for all $z \in Z$. Conversely, if all Φ_z are surjective, then $\Phi(A) \subseteq B$ satisfies the conditions of [48, Proposition C.24], which then implies that $\Phi(A) = B$. \square

Assume now that X is a proper G -space. We proceed with some general constructions of bundles over $G \backslash X$: For this assume that B is any C^* -algebra equipped with an action $\beta : G \rightarrow \text{Aut}(B)$ of G . We define the algebra $C_0(X \times_{G, \beta} B)$ (we shall simply write $C_0(X \times_G B)$ if there is no doubt about the given action) as the set of all bounded continuous functions

$$F : X \rightarrow B \quad \text{such that} \quad F(gx) = \beta_g(F(x))$$

for all $x \in X$ and $g \in G$ and such that the function $x \mapsto \|F(x)\|$ (which is constant on G -orbits) vanishes at infinity on $G \backslash X$. It is easily checked that $C_0(X \times_G B)$ becomes a C^* -algebra when equipped with pointwise addition, multiplication, involution and the sup-norm. Note that this construction is well known under the name of the *generalized fixed point algebras* for the proper action of G on $C_0(X, B)$ (e.g. see [41, 35, 39]).

Lemma 2.2. *$C_0(X \times_G B)$ is the section algebra of a continuous bundle of C^* -algebras over $G \backslash X$ with fibre over the orbit Gx isomorphic to the fixed point algebra B^{G_x} , where $G_x = \{g \in G : gx = x\}$ is the stabilizer of x in G .*

Proof. Assume first that the action of G on X is transitive, i.e., $X = Gx$ for some $x \in X$. Then it is straightforward to check that evaluation at x induces an isomorphism $C_0(Gx \times_G B) \rightarrow B^{G_x}; F \mapsto F(x)$.

For the general case we first note that multiplication with functions in $C_0(G \backslash X)$ provides $C_0(X \times_G B)$ with the structure of a $C_0(G \backslash X)$ -algebra. The ideal $I_{Gx} = C_0((G \backslash X) \setminus \{Gx\})C_0(X \times_G B)$ then coincides with the set of functions $F \in C_0(X \times_G B)$ which vanish on Gx , and hence with the kernel of the restriction map $F \mapsto F|_{Gx}$ from $C_0(X \times_G B)$ into $C_0(Gx \times_G B)$. If we compose this with the evaluation at x we now see that the map $F \mapsto F(x)$ factors through an injective $*$ -homomorphism of the fiber $C_0(X \times_G B)_{Gx}$ into B^{Gx} . We need to show that this map is surjective.

Since images of $*$ -homomorphism between C^* -algebras are closed, it suffices to show that the evaluation map has dense image. For this fix $b \in B^{Gx}$. For any neighborhood U of x choose a positive function $f_U \in C_c(X)$ such that $\text{supp } f_U \subseteq U$ and $\int_G f_U(g^{-1}x) dg = 1$. Then define $F_U \in C_0(X \times_G B)$ by $F_U(y) := \int_G f_U(g^{-1}y) \beta_g(b) dg$ for all $y \in X$. One checks that $F_U \in C_0(X \times_G B)$ and that $F_U(x) \rightarrow b$ as U shrinks to x . This shows the desired density result.

Finally, the fact that the $C_0(X \times_G B)$ is a continuous bundle follows from the fact that the continuous function $x \mapsto \|F(x)\|$ is constant on G -orbits in X , and hence factors through a continuous function on $G \backslash X$. \square

For an induced proper G -space $X = G \times_H Y$ we get the following result.

Lemma 2.3. *Suppose that H is a closed subgroup of G and B is a G -algebra. Then there is a canonical isomorphism*

$$\Phi : C_0((G \times_H Y) \times_G B) \rightarrow C_0(Y \times_H B)$$

given by $\Phi(F)(y) = F([e, y])$ for $F \in C_0((G \times_H Y) \times_G B)$ and $y \in Y$.

Proof. It is straightforward to check that Φ is a well defined $*$ -homomorphism with inverse $\Phi^{-1} : C_0(Y \times_H B) \rightarrow C_0((G \times_H Y) \times_G B)$ given by $\Phi^{-1}(F)([g, y]) = \beta_g(F(y))$, where $\beta : G \rightarrow \text{Aut}(B)$ denotes the given action on B . \square

In some cases the algebra $C_0(X \times_G B)$ has a much easier description.

Definition 2.4. Suppose that G acts on the locally compact space X . A closed subspace $Z \subseteq X$ is called a *topological fundamental domain* for the action of G on X if the mapping $Z \rightarrow G \backslash X; z \mapsto Gz$ is a homeomorphism.

Of course, a topological fundamental domain as in the definition, does not exist in most cases, but the following examples show that there are at least some interesting cases where they do exist:

Example 2.5. For the first example we consider the obvious action of $\text{SO}(n)$ on \mathbb{R}^n . Then the set $Z = \{(x, 0, \dots, 0) : x \geq 0\}$ is a topological fundamental domain for this action.

Example 2.6. For the second example let G (the dihedral group D_4) be generated by a rotation R around the origin in \mathbb{R}^2 by $\frac{\pi}{2}$ radians, and the reflection S through the line $l_S := \{(x, 0) : x \in \mathbb{R}\}$. Writing $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, the group G has the elements $\{E, R, R^2, R^3, S, SR, SR^2, SR^3\}$, where E denotes the unit matrix. Since $G \subseteq \text{GL}(2, \mathbb{Z})$, it's action on \mathbb{R}^2 fixes \mathbb{Z}^2 , and therefore factors through an action on $\mathbb{T}^2 \cong \mathbb{R}^2 / \mathbb{Z}^2$. If we study this action on the fundamental domain $(-\frac{1}{2}, \frac{1}{2}]^2 \subseteq \mathbb{R}^2$ for the translation action of \mathbb{Z}^2 on \mathbb{R}^2 , it is an easy exercise to check that the set

$$Z := \{(e^{2\pi i s}, e^{2\pi i t}) : 0 \leq t \leq \frac{1}{2}, 0 \leq s \leq t\}$$

is a topological fundamental domain for the action of G on \mathbb{T}^2 .

Of course, if we restrict the above action to the subgroup $H := \langle R \rangle \subseteq G$, we obtain an example of a group action with no topological fundamental domain.

Proposition 2.7. *Suppose that $Z \subseteq X$ is a topological fundamental domain for the proper action of G on X . Then, for any G -algebra B , there is an isomorphism*

$$C_0(X \times_G B) \cong \{f : Z \rightarrow B : f(z) \in B^{G_z}\}$$

given by $F \mapsto F|_Z$.

Proof. This is an easy consequence of (the proof of) Lemma 2.2 together with Lemma 2.1. \square

Assume now that we have two commuting actions $\alpha, \beta : G \rightarrow \text{Aut}(B)$ of G on the same C^* -algebra B . Then β induces an action $\tilde{\beta}$ on the crossed product $B \rtimes_\alpha G$ in the canonical way. On the other hand, we also obtain an action $\tilde{\alpha} : G \rightarrow \text{Aut}(C_0(X \times_{G, \beta} B))$ via

$$(\tilde{\alpha}_g(F))(x) = \alpha_g(F(x)).$$

We want to show the following

Proposition 2.8. *In the above situation we have a canonical isomorphism*

$$C_0(X \times_{G, \beta} B) \rtimes_{\tilde{\alpha}} G \cong C_0(X \times_{G, \tilde{\beta}} (B \rtimes_\alpha G)).$$

For the proof we first need

Lemma 2.9. *Let K be a compact group and G a locally compact group such that $\beta : K \rightarrow \text{Aut}(B)$, $\alpha : G \rightarrow \text{Aut}(B)$ are commuting actions of K and G on the C^* -algebra B . Then the fixed-point algebra B^K for the action of K on B is G -invariant and the inclusion $\iota : B^K \rightarrow B$ induces an isomorphism*

$$\iota \rtimes G : B^K \rtimes G \xrightarrow{\cong} (B \rtimes G)^K,$$

where the fixed-point algebra on the right hand side is taken with respect to the action $\tilde{\beta}$ of K on $B \rtimes G$ induced by β in the canonical way.

Proof. Note that the lemma is not obvious, since there might exist G -invariant subalgebras $D \subseteq B$ such that the *full* crossed product $D \rtimes G$ does not include faithfully into $B \rtimes G$ (while this would always be true for the reduced crossed products).

For the proof we use the fact that B^K identifies with the compact operators of a $B \rtimes K$ -Hilbert module defined as follows: we make B into a pre- $B \rtimes K$ Hilbert module, with completion denoted by X_B , by defining the $B \rtimes K$ -valued inner product

$$\langle a, b \rangle_{B \rtimes K} = (k \mapsto \beta_k(a^*)b) \in C(K, B)$$

for $a, b \in X_B$, and right action of $C(K, B) \subseteq B \rtimes K$ on $B \subseteq X_B$ by

$$a \cdot f = \int_K a \beta_k(f(k^{-1})) dk, \quad a \in X_B, f \in C(K, B).$$

We can also check that X_B carries a structure of a *full* left Hilbert- B^K -module given by ${}_{B^K} \langle a, b \rangle = \int_K \beta_k(ab^*) dk$ and a left action of B^K on $B \subseteq X_B$ given by multiplication in B . One easily checks that these Hilbert-module structures on X_B are compatible in the sense that

$${}_{B^K} \langle a, b \rangle c = a \langle b, c \rangle_{B \rtimes K}$$

for all $a, b, c \in X_B$. Thus, the left action of B^K on X_B identifies B^K with $\mathcal{K}(X_B)$. We can then consider the descent module $X_B \rtimes G$, which is a $(B \rtimes K) \rtimes G$ -Hilbert module with $\mathcal{K}(X_B \rtimes G) \cong B^K \rtimes G$ (e.g., see [11] for the formulas for the actions and inner products).

Similarly, via the action of K on $B \rtimes G$ we obtain a $(B \rtimes G) \rtimes K$ -Hilbert bimodule $X_{B \rtimes G}$ with compact operators isomorphic to $(B \rtimes G)^K$. Since the actions of K and G on B commute, we can identify

$$(B \rtimes G) \rtimes K \cong B \rtimes (G \times K) \cong (B \rtimes K) \rtimes G.$$

We now check that under this identification we obtain an isomorphism of right Hilbert $B \rtimes (G \times K)$ -modules between $X_B \rtimes G$ and $X_{B \rtimes G}$, such that $\iota \rtimes G$ intertwines the left actions of $B^K \rtimes G$ and $(B \rtimes G)^K$. Since isomorphisms between Hilbert modules induce isomorphisms between their compact operators, this will imply that $\iota \rtimes G$ is an isomorphism. Note that by construction both modules $X_B \rtimes G$ and $X_{B \rtimes G}$ contain $C_c(G, B)$ as a dense $C_c(G \times K, B)$ -submodule, where we view $C_c(G \times K, B)$ as a dense subalgebra of $B \rtimes (G \times K)$. One then checks that the identity map on $C_c(B \rtimes G)$ induces the desired module isomorphism such that $\iota \rtimes G$ commutes with the left actions of $C_c(G, B^K) = C_c(G, B)^K$ sitting as dense subalgebra in $B^K \rtimes G$ and $(B \rtimes G)^K$, respectively. Thus the identity on $C_c(G, B)$ extends to the desired isomorphism $X_B \rtimes G \cong X_{B \rtimes G}$. \square

Proof of Proposition 2.8. We shall describe the isomorphism via a covariant pair (Φ, U) . For this let $(i_B, i_G) : (B, G) \rightarrow M(B \rtimes_\alpha G)$ denote the canonical inclusions (see [48, Proposition 2.3.4]). We then define

$$\Phi : C_0(X \times_{G, \beta} B) \rightarrow M(C_0(X \times_{G, \tilde{\beta}} (B \rtimes_\alpha G)))$$

by sending $F \in C_0(X \times_{G, \beta} B)$ to the multiplier of $C_0(X \times_{G, \tilde{\beta}} (B \rtimes_\alpha G))$ given by pointwise multiplication with $x \mapsto i_B(F(x))$. Similarly, for $g \in G$ we define $U_g \in M(C_0(X \times_{G, \tilde{\beta}} (B \rtimes_\alpha G)))$ by the pointwise application of $i_G(g)$. One checks that (Φ, U) gives a well defined covariant homomorphism of $(C_0(X \rtimes_{G, \beta} B), G)$ into $M(C_0(X \times_{G, \tilde{\beta}} (B \rtimes_\alpha G)))$ whose integrated form $\Phi \rtimes U$ is $C_0(G \backslash X)$ -linear, since Φ is $C_0(G \backslash X)$ -linear.

Thus it suffices to check that $\Phi \rtimes U$ induces isomorphisms of the fibres. For this fix any $x \in X$. We then obtain a commutative diagram

$$\begin{array}{ccc} C_0(X \times_{G, \beta} B) \rtimes_{\tilde{\alpha}} G & \xrightarrow{\Phi \rtimes U} & C_0(X \times_{G, \tilde{\beta}} (B \rtimes_\alpha G)) \\ \epsilon_x \downarrow & & \downarrow \epsilon_x \\ B^{G_x} \rtimes G & \xrightarrow{i_B \rtimes i_G} & (B \rtimes G)^{G_x} \end{array}$$

and the result follows from Lemma 2.9. \square

We are now going to describe the crossed product $C_0(X) \rtimes G$ in terms of section algebras of suitable C^* -algebra bundles. Consider the algebra $\mathcal{K} = \mathcal{K}(L^2(G))$ equipped with the action $\text{Ad } \rho : G \rightarrow \text{Aut}(\mathcal{K})$, where $\rho : G \rightarrow U(L^2(G))$ denotes the right regular representation of G given by $\rho(g)\xi(t) = \sqrt{\Delta_G(g)}\xi(tg)$ for $\xi \in L^2(G)$. Let $\text{lt}, \text{rt} : G \rightarrow \text{Aut}(C_0(G))$ denote the actions given by left and right translation on G , respectively. It then follows from the extended version of the Stone-von Neumann theorem (see [48, Theorem 4.24], but see [46] or a more direct proof) that

$$(2.1) \quad M \times \lambda : C_0(G) \rtimes_{\text{lt}} G \xrightarrow{\cong} \mathcal{K}(L^2(G)),$$

where $M \times \lambda$ is the integrated form of the covariant pair (M, λ) with $M : C_0(G) \rightarrow B(L^2(G))$ being the representation by multiplication operators and $\lambda : G \rightarrow U(L^2(G))$ the left regular representation of G . Let $\tilde{\text{rt}} : G \rightarrow \text{Aut}(C_0(G) \rtimes_{\text{lt}} G)$ denote the action induced from the right translation action $\text{rt} : G \rightarrow \text{Aut}(C_0(G))$. For $f \in C_0(G)$ one checks that $M(\text{rt}_g(f)) = \rho(g)M(f)\rho(g^{-1})$. From this and the fact that ρ commutes with λ it follows that

$$M \times \lambda(\tilde{\text{rt}}_g(\varphi)) = \rho(g)(M \times \lambda(\varphi))\rho(g^{-1})$$

for all $g \in G$ and $\varphi \in C_0(G) \rtimes_{\text{lt}} G$. We shall use all this for the proof of following result, where we write \mathcal{K} for $\mathcal{K}(L^2(G))$:

Theorem 2.10. *Let G be a locally compact group acting properly on the locally compact space X with corresponding action $\tau : G \rightarrow \text{Aut}(C_0(X))$ and let $\beta : G \rightarrow \text{Aut}(B)$ be any action of G on a C^* -algebra B . Then there is a canonical isomorphism*

$$(C_0(X) \otimes B) \rtimes_{\tau \otimes \beta} G \cong C_0(X \times_{G, \text{Ad } \rho \otimes \beta} (\mathcal{K} \otimes B)).$$

Proof. We consider the commuting actions $\text{rt} \otimes \text{id}_B$ and $\text{lt} \otimes \beta$ of G on $C_0(G, B) = C_0(G) \otimes B$. It follows from Proposition 2.8 that there is a canonical isomorphism

$$C_0(X \times_{G, \text{rt} \otimes \text{id}_B} (C_0(G, B) \rtimes_{\text{lt} \otimes \beta} G)) \cong C_0(X \times_{G, \text{rt} \otimes \text{id}_B} C_0(G, B)) \rtimes_{\text{lt} \otimes \beta} G.$$

If we define $X \times_{G, \text{rt}} G = G \backslash (X \times G)$ with respect to the action $g(x, h) = (gx, hg^{-1})$, and if we equip this space with the G -action given by left translation on the second factor, called $\tilde{\text{lt}}$ below, we obtain a $\tilde{\text{lt}} \otimes \beta - \text{lt} \otimes \beta$ -equivariant isomorphism

$$C_0(X \times_{G, \text{rt}} G) \otimes B \cong C_0(X \times_{G, \text{rt} \otimes \text{id}_B} C_0(G, B)),$$

which induces an isomorphism of the respective crossed-products. We further observe that the map $X \times_{G, \text{rt}} G \rightarrow X; [x, g] \mapsto gx$ is a homeomorphism (with inverse given by $x \mapsto [x, e]$) which transforms $\tilde{\text{lt}}$ into the given action τ on X . Combining all this, we obtain a canonical isomorphism

$$C_0(X \times_{G, \text{rt} \otimes \text{id}_B} C_0(G, B)) \rtimes_{\tilde{\text{lt}} \otimes \beta} G \cong C_0(X, B) \rtimes_{\tau \otimes \beta} G.$$

Using the isomorphism $C_0(X \times_{G, \text{rt} \otimes \text{id}_B} C_0(G, B)) \rtimes_{\tilde{\text{lt}} \otimes \beta} G \cong C_0(X \times_{G, \text{rt} \otimes \text{id}_B} (C_0(G, B) \rtimes_{\text{lt} \otimes \beta} G))$, all remains to do is to identify $C_0(G, B) \rtimes_{\text{lt} \otimes \beta} G$ with $\mathcal{K}(L^2(G)) \otimes B$ equivariantly with respect to the action $\text{rt} \otimes \text{id}_B$ and $\text{Ad } \rho \otimes \beta$. But such isomorphism is well known from Takesaki-Takai duality: it is straightforward to check that the isomorphism $\Phi : C_0(G, B) \rightarrow C_0(G, B)$ given by $\Phi(f)(g) = \beta_{g^{-1}}(f(g))$ transforms the action $\text{lt} \otimes \beta$ to the action $\text{lt} \otimes \text{id}_B$ and the action $\text{rt} \otimes \text{id}_B$ to the action $\text{rt} \otimes \beta$. This induces an isomorphism

$$\begin{aligned} C_0(X \times_{G, \text{rt} \otimes \text{id}_B} (C_0(G, B) \rtimes_{\text{lt} \otimes \beta} G)) &\cong C_0(X \times_{G, \text{rt} \otimes \beta} (C_0(G) \rtimes_{\text{lt}} G) \otimes B) \\ &\cong C_0(X \times_{G, \text{Ad } \rho \otimes \beta} (\mathcal{K} \otimes B)). \end{aligned}$$

□

Corollary 2.11. *Let G be a locally compact group acting properly on the locally compact space X and let $\mathcal{K} = \mathcal{K}(L^2(G))$. Then*

$$C_0(X) \rtimes_{\tau} G \cong C_0(X \times_{G, \text{Ad } \rho} \mathcal{K}).$$

Remark 2.12. Explicitly, for any orbit $Gx \in G \backslash X$ the evaluation map $q_x : C_0(X \times_G \mathcal{K}) \rightarrow \mathcal{K}^{G_x}$ can be described on the level of $C_0(X) \rtimes G$ by the composition of maps

$$C_0(X) \rtimes G \xrightarrow{q_{Gx}} C_0(Gx) \rtimes G \cong C_0(G/G_x) \rtimes G \xrightarrow{M \times \lambda} \mathcal{K}^{G_x},$$

where the first map is induced by the G -equivariant restriction map $C_0(X) \rightarrow C_0(Gx)$ and the second is induced by the G -homeomorphism $G/G_x \rightarrow Gx; gG_x \mapsto gx$.

Example 2.13. Let $X := \mathbb{T}$, $G := \mathbb{Z}/2$ acting by conjugation on the circle. Then

$$C(\mathbb{T}) \rtimes \mathbb{Z}/2 \cong \{f \in C([0, 1], M_2(\mathbb{C})) \mid f(0) \text{ and } f(1) \text{ are diagonal}\}.$$

This is immediate from Corollary 2.11, using the basis $\{\frac{1}{\sqrt{2}}(1, 1), \frac{1}{\sqrt{2}}(1, -1)\}$ for $\ell^2(\mathbb{Z}/2)$ to identify it with \mathbb{C}^2 (this diagonalizes the $\mathbb{Z}/2$ -action.) A continuous function $f : \mathbb{T} \rightarrow \mathcal{K}(\ell^2(\mathbb{Z}/2))$ such that $f(gx) = \text{Ad } g(f(x))$ is determined by its restriction to $\{z \in \mathbb{T} \mid \text{Im}(z) \geq 0\}$, where, using the above basis, we can identify it

with a map $f: [0, 1] \rightarrow M_2(\mathbb{C})$ such that $f(0)$ and $f(1)$ commute with the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. The commutant of this matrix consists of the diagonal matrices.

Theorem 2.10 can be extended to the case of proper actions on general C^* -algebras B , *i.e.* such that B is an $X \rtimes G$ -algebra for some proper G -space X . Thus, B is a $C_0(X)$ -algebra equipped with a G -action $\beta: G \rightarrow \text{Aut}(B)$ such that the structure map $\phi: C_0(X) \rightarrow ZM(B)$ is G -equivariant. In this situation the generalized fixed point algebra $B^{G,\beta}$ can be constructed as follows: we consider the algebra $C_0(X \times_G B)$ as studied above. If $b \in B$, we write $b(y)$ for the evaluation of B in the fiber B_y , $y \in X$. Similarly, for $F \in C_0(X \times_G B)$ we write $F(x, y)$ for the evaluation of the element $F(x) \in B$ in the fiber B_y . Then $C_0(X \times_G B)$ becomes a $C_0(G \backslash (X \times X))$ -algebra via the structure map

$$\Phi: C_0(G \backslash (X \times X)) \rightarrow ZM(C_0(X \times_G B)); (\Phi(\varphi)F)(x, y) = \varphi([x, y])F(x, y).$$

We then define $B^{G,\beta}$ (or just B^G if β is understood) as the restriction of $C_0(X \times_G B)$ to $G \backslash \Delta(X) \cong G \backslash X$, where $\Delta(X) = \{(x, x) : x \in X\}$. Note that with this notation we have $C_0(X \times_{G,\beta} B) \cong (C_0(X) \otimes B)^{G,\tau \otimes \beta}$ (if τ denotes the corresponding action on $C_0(X)$). Moreover, if G is compact, this notation coincides with the usual notation of the fixed-point algebra B^G .

With this notation we get

Theorem 2.14. *Suppose that B is an $X \rtimes G$ -algebra for the proper G -space X via some action $\beta: G \rightarrow \text{Aut}(B)$. Then $B \rtimes G$ is isomorphic to $(\mathcal{K} \otimes B)^{G, \text{Ad } \rho \otimes \beta}$.*

Proof. Consider the $C_0(X \times X)$ -algebra $C_0(X, B)$. By Theorem 2.10 we have $C_0(X, B) \rtimes_{\tau \otimes \beta} G \cong C_0(X \times_{G, \text{Ad } \rho \otimes \beta} (\mathcal{K} \otimes B))$. The crossed product $C_0(X, B) \rtimes G$ carries a canonical structure as a $C_0(G \backslash (X \times X))$ -algebra which is induced from the $C_0(X \times X)$ -structure of $C_0(X, B)$. A careful look at the proof of the isomorphism $C_0(X, B) \rtimes G \cong C_0(X \times_G (\mathcal{K} \otimes B))$ reveals that this isomorphism preserves the $C_0(G \backslash (X \times X))$ -structures on both algebras. Using the G -isomorphism $C_0(X, B)|_{\Delta(X)} \cong B$ which is induced from the $*$ -homomorphism $C_0(X, B) \cong C_0(X) \otimes B \rightarrow B; (f \otimes b) \mapsto fb$, we now obtain a chain of isomorphisms

$$\begin{aligned} B \rtimes G &\cong (C_0(X, B)|_{\Delta(X)}) \rtimes G \\ &\cong (C_0(X, B) \rtimes G)|_{G \backslash \Delta(X)} \\ &\cong (C_0(X \times_G (\mathcal{K} \otimes B)))|_{G \backslash \Delta(X)} \\ &= (\mathcal{K} \otimes B)^{G, \text{Ad } \rho \otimes \beta}. \end{aligned}$$

□

Remark 2.15. Using this, it is not difficult to check that for any $X \rtimes G$ -algebra B for some proper G -space X the crossed product $B \rtimes G$ is a $C_0(G \backslash X)$ -algebra with fiber at an orbit Gx isomorphic to $(\mathcal{K} \otimes B_x)^{G_x, \text{Ad } \rho \otimes \beta^x}$, where $\beta^x: G_x \rightarrow \text{Aut}(B_x)$ is the action induced from β in the canonical way.

The results of this section fit into the framework of generalized fixed-point algebras (see the work of Marc Rieffel and also Ralf Meyer in [43, 44, 39]); our aim here is not generality, but explicitness, and we have taken a direct approach.

3. THE MACKEY-RIEFFEL-GREEN THEOREM FOR PROPER ACTIONS

The Mackey-Rieffel-Green theorem (or Mackey-Rieffel-Green machine) supplies, under some suitable conditions on a given C^* -dynamical system (A, G, α) , a systematic way of describing the irreducible representations (or primitive ideals) of the crossed product $A \rtimes_\alpha G$ in terms of the associated action of G on $\text{Prim}(A)$ by

inducing representations (or ideals) from the stabilizers for this action. We refer to [48, 20] for recent discussions of this general machinery, and to [30, 31] for some important contributions towards this theory.

In this section we want to give a self-contained exposition of the Mackey-Rieffel-Green machine in the special case of crossed products by proper actions on spaces, in which the result will follow easily from the bundle description of the crossed product $C_0(X) \rtimes G$ as obtained in the previous section and the following explicit description of the fibers. This explicit description of the fibers will also play an important rôle in our description of the Fell topology on $(C_0(X) \rtimes G)^\wedge$ as given in §4 below.

From Corollary 2.11 and Lemma 2.2, if G acts properly on X , then the fibre $(C_0(X) \rtimes G)_{G(x)}$ of the crossed product $C_0(X) \rtimes G \cong C_0(X \times_{G, \text{Ad } \rho} \mathcal{K}(L^2(G)))$ at the orbit $G(x)$ is isomorphic to $C_0(G/G_x) \rtimes G$, equivalently, to the fixed-point algebra $\mathcal{K}(L^2(G))^{G_x, \text{Ad } \rho}$ for the compact stabilizer G_x at x , where $\rho : G \rightarrow U(L^2(G))$ denotes the right regular representation of G . We now analyse the structure of this fibre, using the Peter-Weyl theorem.

Let us recall some basic constructions in representation theory. If \mathcal{H} is any Hilbert space we denote by \mathcal{H}^* its *adjoint Hilbert space*, that is

$$\mathcal{H}^* = \{\xi^* : \xi \in \mathcal{H}\}$$

with the linear operations $\lambda\xi^* + \mu\eta^* = (\bar{\lambda}\xi + \bar{\mu}\eta)^*$ and the inner product $\langle \xi^*, \eta^* \rangle = \langle \eta, \xi \rangle$. Note that \mathcal{H}^* identifies canonically with the space of continuous linear functionals on \mathcal{H} . If $\sigma : G \rightarrow \mathcal{U}(\mathcal{H})$ is a representation of the group G on the Hilbert space \mathcal{H} , then its *adjoint representation* $\sigma^* : G \rightarrow \mathcal{U}(\mathcal{H}^*)$ is given by

$$\sigma^*(g)\xi^* := (\sigma(g)\xi)^*.$$

Assume now that K is a compact subgroup of G and let $\sigma : K \rightarrow U(V_\sigma)$ be a unitary representation of K . We then define a representation $\pi^\sigma = P^\sigma \times U^\sigma$ of the crossed product $C_0(G/K) \rtimes G$ as follows: we define

$$(3.1) \quad \mathcal{H}_{U^\sigma} := \{\xi \in L^2(G, V_\sigma) : \xi(gk) = \sigma(k^{-1})\xi(g) \forall g \in G, k \in K\}.$$

Then define the covariant representation (P^σ, U^σ) of $C_0(G/K) \rtimes G$ on \mathcal{H}_{U^σ} by

$$(3.2) \quad (P^\sigma(\varphi)\xi)(g) = \varphi(gK)\xi(g) \quad \text{and} \quad (U^\sigma(t)\xi)(g) = \xi(t^{-1}g).$$

Note that in classical representation theory of locally compact groups the covariant pair (P^σ, U^σ) is often called the “system of imprimitivity” induced from the representation σ of K (e.g. see [10]).

In what follows, if $\sigma \in \widehat{K}$, we denote by $p_\sigma \in C^*(K)$ the central projection corresponding to σ , i.e., we have $p_{\sigma^*} = d_\sigma \chi_{\sigma^*}$, where $\chi_{\sigma^*}(k) = \text{trace } \sigma^*(k)$ denotes the character of the adjoint σ^* of σ and d_σ denotes the dimension of V_σ . Note that it follows from the Peter-Weyl theorem (e.g. see [13, Chapter 7]) that $\sigma(p_\sigma) = 1_{V_\sigma}$ and $\tau(p_\sigma) = 0$ for all $\tau \in \widehat{K}$ not equivalent to σ , and that $\sum_{\sigma \in \widehat{K}} p_\sigma$ converges strictly to the unit $1 \in M(C^*(K))$.

Lemma 3.1. *Let G be a locally compact group and let K be a compact subgroup of G acting on $\mathcal{K} = \mathcal{K}(L^2(G))$ via $k \mapsto \text{Ad } \rho(k)$. For each $\sigma \in \widehat{K}$ let $L^2(G)_\sigma := \rho(p_{\sigma^*})L^2(G)$. Then the following are true:*

- (i) $L^2(G) = \bigoplus_{\sigma \in \widehat{K}} L^2(G)_\sigma$;
- (ii) each space $L^2(G)_\sigma$ is $\rho(K)$ -invariant and decomposes into a tensor product $\mathcal{H}_{U^\sigma} \otimes V_\sigma^*$ such that $\rho(k)|_{L^2(G)_\sigma} = 1_{\mathcal{H}_{U^\sigma}} \otimes \sigma^*(k)$ for all $k \in K$;
- (iii) $\mathcal{K}^K \cong \bigoplus_{\sigma \in \widehat{K}} \mathcal{K}(\mathcal{H}_{U^\sigma})$, where the isomorphism is given by sending an operator $T \in \mathcal{K}(\mathcal{H}_{U^\sigma})$ to the operator $T \otimes 1_{V_\sigma^*} \in \mathcal{K}(L^2(G)_\sigma)$ under the decomposition of (ii);

- (iv) the projection of $C_0(G/K) \rtimes G \cong \mathcal{K}^K$ onto the factor $\mathcal{K}(\mathcal{H}_{U^\sigma})$ in the decomposition in (iii) is equal to the representation $\pi^\sigma = P^\sigma \times U^\sigma$ constructed above in (3.1) and (3.2).

Proof. The proof is basically a consequence of the Peter-Weyl Theorem for the compact group K . Item (i) follows from the fact that the central projections p_{σ^*} add up to the unit in $M(C^*(K))$ with respect to the strict topology.

For the proof of (ii) we first consider the induced G -representation U^{λ_K} , where λ_K denotes the left regular representation of K . It acts on the Hilbert space

$$\mathcal{H}_{U^\lambda} := \{\xi \in L^2(G, L^2(K)) : \xi(gk, l) = \xi(g, kl) \forall g \in G, k, l \in K\}.$$

But a short computation shows that $U^{\lambda_K} \cong \lambda_G$, the left regular representation of G , where a unitary intertwining operator is given by

$$\Phi_\lambda : \mathcal{H}_{U^\lambda} \xrightarrow{\cong} L^2(G); \Phi_\lambda(\xi)(g) = \xi(g, e).$$

By the Peter-Weyl theorem we know that $L^2(K)$ decomposes into the direct sum $\bigoplus_{\sigma \in \widehat{K}} V_\sigma \otimes V_\sigma^*$ in such a way that the left regular representation decomposes as $\lambda_K \cong \bigoplus_{\sigma} \sigma \otimes 1_{V_\sigma^*}$ and the right regular representation decomposes as $\rho_K \cong \bigoplus_{\sigma} 1_{V_\sigma} \otimes \sigma^*$. (e.g. see [13, Theorem 7.2.3] together with the obvious isomorphism $V_\sigma \otimes V_\sigma^* \cong \text{End}(V_\sigma)$) This induces a decomposition

$$(3.3) \quad L^2(G) \cong \mathcal{H}_{U^\lambda} \cong \bigoplus_{\sigma \in \widehat{K}} \mathcal{H}_{U^\sigma} \otimes V_\sigma^*.$$

To see how the isomorphism Φ_λ restricts to the direct summand $\mathcal{H}_{U^\sigma} \otimes V_\sigma^*$ we should note that the inclusion of $V_\sigma \otimes V_\sigma^*$ into $L^2(K)$ is given by sending an elementary vector $v \otimes w^*$ to the function $k \mapsto \sqrt{d_\sigma} \langle \sigma(k^{-1})v, w \rangle$. The corresponding inclusion of $\mathcal{H}_{U^\sigma} \otimes V_\sigma^*$ into $L^2(G)$ is therefore given by sending an elementary vector $\xi \otimes w^*$ to the L^2 -function $g \mapsto \sqrt{d_\sigma} \langle \xi(g), w \rangle$. One can easily check directly, using the orthogonality relations for matrix coefficients on K , that this defines an isometry Φ_σ from $\mathcal{H}_{U^\sigma} \otimes V_\sigma^*$ into $L^2(G)_\sigma$. To show that the image lies in $L^2(G)_\sigma$ we compute

$$\begin{aligned} (\rho(p_{\sigma^*})\Phi_\sigma(\xi \otimes v^*))(g) &= \int_K p_{\sigma^*}(k) \sqrt{d_\sigma} \langle \xi(gk), v \rangle dk \\ &= \int_K p_{\sigma^*}(k) \sqrt{d_\sigma} \langle \xi(g), \sigma(k)v \rangle dk \\ &= \sqrt{d_\sigma} \langle \xi(g), v \rangle = \Phi_\sigma(\xi \otimes v^*)(g) \end{aligned}$$

which proves the claim. Using (i) and (3.3), this also implies that the image is all of $L^2(G)_\sigma$. One easily checks that Φ_σ intertwines the representation $1 \otimes \sigma^*$ on $\mathcal{H}_{U^\sigma} \otimes V_\sigma^*$ with the restriction of ρ to K .

For the proof of (iii) we first observe that $T \in \mathcal{K}(L^2(G))^K$ if and only if T commutes with $\rho(k)$ for all $k \in K$, which then implies, via integration, that T commutes with $\rho(p_\sigma)$ for all $\sigma \in \widehat{K}$. It follows that \mathcal{K}^K lies in $\bigoplus_{\sigma \in \widehat{K}} \mathcal{K}(L^2(G)_\sigma) \subseteq \mathcal{K}(L^2(G))$. Now, using the decomposition $L^2(G)_\sigma = \mathcal{H}_{U^\sigma} \otimes V_\sigma^*$ as in (ii) we get

$$\mathcal{K}(L^2(G)_\sigma)^{\text{Ad } \rho(K)} \cong (\mathcal{K}(\mathcal{H}_{U^\sigma}) \otimes \mathcal{K}(V_\sigma^*))^{\text{Id} \otimes \text{Ad } \sigma^*(K)} = \mathcal{K}(\mathcal{H}_{U^\sigma}) \otimes \mathbb{C}1_{V_\sigma^*}.$$

Finally, item (iv) now follows from the fact that the restriction of the representation $M \times \lambda : C_0(G/K) \rtimes G \xrightarrow{\cong} \mathcal{K}(L^2(G))^K$ to the subspace $\mathcal{H}_{U^\sigma} \otimes V_\sigma^*$ clearly coincides with $(P^\sigma \times U^\sigma) \otimes 1_{V_\sigma^*}$. \square

Example 3.2. The above decomposition becomes easier in the case where the compact subgroup K of G is abelian, since in this case the irreducible representations σ of K are one-dimensional. We therefore get $p_\sigma = \bar{\sigma}$ viewed as an element of

$C(K) \subseteq C^*(K)$ and then, for $\xi \in L^2(G)$ we have $\xi \in L^2(G)_\sigma = \rho(p_\sigma)L^2(G)$ if and only if

$$\xi(g) = (\rho(p_\sigma)\xi)(g) = \int_K \bar{\sigma}(k)\xi(gk) dk$$

for almost all $g \in G$. For $l \in K$ we then get

$$\begin{aligned} \xi(gl) &= \int_K \bar{\sigma}(k)\xi(glk) dk \stackrel{k \mapsto l^{-1}k}{=} \int_K \bar{\sigma}(l^{-1}k)\xi(gk) dk \\ &= \sigma(l)\xi(g), \end{aligned}$$

which shows that in this situation we have $L^2(G)_\sigma = \mathcal{H}_{U^\sigma}$ for all $\sigma \in \widehat{K}$. Thus we get the direct decompositions

$$L^2(G) = \bigoplus_{\sigma \in \widehat{K}} \mathcal{H}_{U^\sigma} \quad \text{and} \quad \mathcal{K}(L^2(G))^{K, \text{Ad } \rho} = \bigoplus_{\sigma \in \widehat{K}} \mathcal{K}(\mathcal{H}_{U^\sigma}).$$

This picture becomes even more transparent if G happens also to be abelian. In that case one checks that the Fourier transform $\mathcal{F} : L^2(G) \rightarrow L^2(\widehat{G})$ maps the subspace $L^2(G)_\sigma$ of $L^2(G)$ to the subspace $L^2(\widehat{G}_\sigma)$ of $L^2(\widehat{G})$ in which

$$\widehat{G}_\sigma := \{\chi \in \widehat{G} : \chi|_K = \sigma\}$$

(we leave the details as an exercise to the reader). In the special case $G = \mathbb{T}$ and $K = \mathbb{C}(n)$, the group of all n th roots of unity, we get $\widehat{\mathbb{C}(n)} \cong \mathbb{Z}/n\mathbb{Z}$ and, using Fourier transform, the composition of Lemma 3.1 becomes

$$L^2(\mathbb{T}) \cong \ell^2(\mathbb{Z}) \cong \bigoplus_{[l] \in \mathbb{Z}/n\mathbb{Z}} \ell^2(l\mathbb{Z}) \quad \text{and} \quad \mathcal{K}(L^2(\mathbb{T}))^{C(n)} \cong \bigoplus_{[l] \in \mathbb{Z}/n\mathbb{Z}} \mathcal{K}(\ell^2(l\mathbb{Z})).$$

If $L \subseteq K$ are two compact subgroups of the locally compact group G , then we certainly have $\mathcal{K}(L^2(G))^{\text{Ad } \rho(K)} \subseteq \mathcal{K}(L^2(G))^{\text{Ad } \rho(L)}$. For later use, it is important for us to have a precise understanding of this inclusion. In the following lemma we denote by $\text{Rep}(L)$ the equivalence classes of **all** unitary representations of a group L and $\text{Rep}(A)$ denotes the equivalence classes of all non-degenerate $*$ -representation of a C^* -algebra A . In this notation we obtain a map

$$\text{Ind}_L^G : \text{Rep}(L) \rightarrow \text{Rep}(C_0(G/L) \rtimes G); \sigma \mapsto \text{Ind}_L^G \sigma := P^\sigma \times U^\sigma,$$

and similarly for K , with P^σ and U^σ defined as in (3.1) and (3.2). Moreover, induction of unitary representations gives a mapping $\text{Ind}_L^K : \text{Rep}(L) \rightarrow \text{Rep}(K)$ and the inclusion $C_0(G/K) \rtimes G$ into $C_0(G/L) \rtimes G$ induced by the obvious inclusion of $C_0(G/K)$ into $C_0(G/L)$ induces a mapping

$$\text{Res}_{G/L}^{G/K} : \text{Rep}(C_0(G/L) \rtimes G) \rightarrow \text{Rep}(C_0(G/K) \rtimes G).$$

Lemma 3.3. *Suppose that $L \subseteq K$ and G are as above. Then the diagram*

$$(3.4) \quad \begin{array}{ccc} \text{Rep}(K) & \xrightarrow{\text{Ind}_K^G} & \text{Rep}(C_0(G/K) \rtimes G) \\ \text{Ind}_L^K \uparrow & & \uparrow \text{Res}_{C_0(G/L)}^{C_0(G/K)} \\ \text{Rep}(L) & \xrightarrow{\text{Ind}_L^G} & \text{Rep}(C_0(G/L) \rtimes G) \end{array}$$

commutes.

Proof. Let $\sigma \in \text{Rep}(L)$. Recall from (3.1) that the Hilbert space \mathcal{H}_{U^σ} for the induced representation $\text{Ind}_L^G \sigma = P^\sigma \times U^\sigma$ is defined as

$$\mathcal{H}_{U^\sigma} = \{\xi \in L^2(G, V_\sigma) : \xi(gl) = \sigma(l^{-1})\xi(g) \ \forall g \in G, l \in L\},$$

and similar constructions give the Hilbert spaces for the representations $\text{Ind}_L^K \sigma$ and $\text{Ind}_K^G \tau$ for some $\tau \in \text{Rep}(K)$. In particular, for $\tau := \text{Ind}_L^K \sigma$, we deduce the formula

$$\begin{aligned} \mathcal{H}_{U^\tau} &= \{\eta \in L^2(G, L^2(K, V_\sigma)) : \eta(gk, h) = \eta(g, kh) \\ &\quad \text{and } \eta(g, hl) = \sigma(l^{-1})\eta(g, h) \ \forall g \in G, k, h \in K, l \in L\}. \end{aligned}$$

It is then straightforward to check that the operator

$$V : \mathcal{H}_{U^\sigma} \rightarrow \mathcal{H}_{U^\tau}; \ (V\xi)(g, h) = \xi(gh)$$

is a unitary with inverse V^{-1} given by $(V^{-1}\eta)(g) = \eta(g, e)$ such that V intertwines the G -representations U^σ and U^τ . Moreover, for any $\varphi \in C_0(G/K)$ we get

$$\begin{aligned} (P^\tau(\varphi)V\xi)(g, h) &= \varphi(gK)(V\xi)(g, h) = \varphi(gK)\xi(gh) = \varphi(ghK)\xi(gh) \\ &= (P^\sigma(\varphi)\xi)(gh) = (VP^\sigma(\varphi)\xi)(g, h). \end{aligned}$$

This proves that that V intertwines $\text{Res}_{G/L}^{G/K} \circ \text{Ind}_L^G \sigma$ with $\text{Ind}_K^L \tau = \text{Ind}_K^G \circ \text{Ind}_L^K \sigma$. \square

By Lemma 3.1 we have isomorphisms

$$C_0(G/K) \rtimes G \stackrel{M \rtimes \lambda}{\cong} \mathcal{K}(L^2(G))^{\text{Ad } \rho(K)} = \bigoplus_{\tau \in \widehat{K}} \mathcal{K}(\mathcal{H}_{U^\tau}) \otimes 1_{V_\tau^*}$$

where the right equation is induced by the decomposition

$$L^2(G) = \bigoplus_{\tau \in \widehat{K}} \mathcal{H}_{U^\tau} \otimes V_\tau^*.$$

Thus we see that, as a subalgebra of $\mathcal{K}(L^2(G))$, the algebra $\mathcal{K}(L^2(G))^{\text{Ad } \rho(K)}$ decomposes in blocks of compact operators $\mathcal{K}(\mathcal{H}_{U^\tau})$ such that each block $\mathcal{K}(\mathcal{H}_{U^\tau})$ appears with the multiplicity $\dim V_\tau$ in this decomposition. If $L \subseteq K$, we get a similar decomposition of $\mathcal{K}(L^2(G))^{\text{Ad } \rho(L)}$ indexed over all $\sigma \in \widehat{L}$ with multiplicities $\dim V_\sigma$. Since $L \subseteq K$ we get $\mathcal{K}(L^2(G))^{\text{Ad } \rho(K)} \subseteq \mathcal{K}(L^2(G))^{\text{Ad } \rho(L)}$. To understand this inclusion, we need to know how many copies of each block $\mathcal{K}(\mathcal{H}_{U^\tau})$, $\tau \in \widehat{K}$, appear in any given block $\mathcal{K}(\mathcal{H}_{U^\sigma})$, $\sigma \in \widehat{L}$, of the algebra $\mathcal{K}(L^2(G))^{\text{Ad } \rho(L)}$. This multiplicity number m_τ^σ clearly coincides with the multiplicity of the representation $\pi^\tau = P^\tau \times U^\tau$ of $C_0(G/K) \rtimes G \cong \mathcal{K}(L^2(G))^{\text{Ad } \rho(K)}$ in the restriction of the representation $\pi^\sigma = P^\sigma \times U^\sigma$ of $C_0(G/L) \rtimes G \cong \mathcal{K}(L^2(G))^{\text{Ad } \rho(L)}$ to the subalgebra $C_0(G/K) \rtimes G$. By the above lemma, this multiplicity coincides with the multiplicity of τ in the induced representation $\text{Ind}_L^K \sigma$, and by Frobenius reciprocity, this equals the multiplicity of σ in the restriction $\tau|_L$. Thus we conclude

Proposition 3.4. *Suppose that $L \subseteq K$ are two compact subgroups of the locally compact group G . For each $\tau \in \widehat{K}$ and $\sigma \in \widehat{L}$ let m_τ^σ denote the multiplicity of σ in the restriction $\tau|_L$. Then, under the inclusion $\mathcal{K}(L^2(G))^{\text{Ad } \rho(K)} \subseteq \mathcal{K}(L^2(G))^{\text{Ad } \rho(L)}$ each block $\mathcal{K}(\mathcal{H}_{U^\tau})$ of $\mathcal{K}(L^2(G))^{\text{Ad } \rho(K)}$ appears with multiplicity m_τ^σ in each block $\mathcal{K}(\mathcal{H}_{U^\sigma})$ of $\mathcal{K}(L^2(G))^{\text{Ad } \rho(L)}$.*

We should note that since each block $\mathcal{K}(\mathcal{H}_{U^\sigma})$ appears with multiplicity $\dim V_\sigma$ in the representation of $\mathcal{K}(L^2(G))^{\text{Ad } \rho(L)}$ on $L^2(G)$, and similarly for $\tau \in \widehat{K}$, we get the equation

$$\dim V_\tau = \sum_{\sigma \in \widehat{L}} m_\tau^\sigma \dim V_\sigma,$$

for the total multiplicity $\dim V_\tau$ of $\mathcal{K}(\mathcal{H}_{U^\tau})$ in $\mathcal{K}(L^2(G))^{\text{Ad } \rho(K)}$. Let us illustrate the above results in a concrete example:

Example 3.5. Let us consider the action of the finite group $G = D_4 = \langle R, S \rangle \subseteq \text{GL}(2, \mathbb{Z})$ with $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on \mathbb{T}^2 as described in Example 2.6. It was shown in that example that we have the topological fundamental domain

$$Z := \{(e^{2\pi i s}, e^{2\pi i t}) : 0 \leq t \leq \frac{1}{2}, 0 \leq s \leq t\}$$

and it follows then from Proposition 2.7 that the crossed product $C_0(\mathbb{T}^2) \rtimes G$ is isomorphic to the sub-homogeneous algebra

$$A := \{f \in C(Z, \mathcal{K}(\ell^2(G))) : f(z, w) \in \mathcal{K}(\ell^2(G))^{\text{Ad } \rho(G_{(z,w)})}\},$$

where $G_{(z,w)}$ denotes the stabilizer of the point (z, w) under the action of G . In what follows we will identify Z with the triangle $\{(s, t) \in \mathbb{R}^2 : 0 \leq t \leq \frac{1}{2}, 0 \leq s \leq t\}$ and we write $G_{(s,t)}$ for the corresponding stabilizers of the points $(e^{2\pi i s}, e^{2\pi i t})$. A straightforward computation shows that

- $G_{(s,t)} = \{E\}$ if $0 < t < \frac{1}{2}, 0 < s < t$,
- $G_{(s,s)} = \langle RS \rangle =: K_1$ if $0 < s < \frac{1}{2}$,
- $G_{(0,t)} = \langle R^2 S \rangle =: K_2$ if $0 < t < \frac{1}{2}$,
- $G_{(s, \frac{1}{2})} = \langle S \rangle =: K_3$ if $0 < s < \frac{1}{2}$,
- $G_{(0, \frac{1}{2})} = \langle S, R^2 \rangle =: H$, and
- $G_{(0,0)} = G_{(\frac{1}{2}, \frac{1}{2})} = G$.

It follows that $\mathcal{K}(\ell^2(G))^{G_{(s,t)}} = \mathcal{K}(\ell^2(G)) \cong M_8(\mathbb{C})$ whenever $0 < t < \frac{1}{2}, 0 < s < t$. In the three cases where $G_{(s,t)} = K_i$, $i = 1, 2, 3$, is a subgroup of order two, we get two one-dimensional representations $\{1_{K_i}, \epsilon_{K_i}\}$, $i = 1, 2, 3$, so by choosing suitable bases of $\ell^2(G)$, in each case the algebra $\mathcal{K}(\ell^2(G))^{G_{(s,t)}}$ has the form

$$\begin{pmatrix} A_{1_{K_i}} & 0 \\ 0 & A_{\epsilon_{K_i}} \end{pmatrix} \quad A_{1_{K_i}}, A_{\epsilon_{K_i}} \in M_4(\mathbb{C}),$$

where the 4×4 -blocks act on the four-dimensional subspaces $\rho(p_{1_{K_i}})\ell^2(G)$ and $\rho(p_{\epsilon_{K_i}})\ell^2(G)$, respectively, where ρ denotes the right regular representation of G restricted to the respective stabilizer. We refer to the discussion before Lemma 3.1 for the definition of $p_{1_{K_i}}$ and $p_{\epsilon_{K_i}}$.

At the corner $(0, \frac{1}{2})$ we have the stabilizer $H = \langle S, R^2 \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, so we get four one-dimensional representations $1, \mu_1, \mu_2, \mu_3$ of this group given by

$$\mu_1(R^2) = -\mu_1(S) = 1, \quad \mu_2(R^2) = \mu_2(S) = -1 \quad \text{and} \quad \mu_3(R^2) = -\mu_3(S) = -1.$$

Therefore $\mathcal{K}(\ell^2(G))^{G_{(0, \frac{1}{2})}}$ decomposes as

$$\begin{pmatrix} B_1 & & & \\ & B_{\mu_1} & & \\ & & B_{\mu_2} & \\ & & & B_{\mu_3} \end{pmatrix} \quad B_1, B_{\mu_1}, B_{\mu_2}, B_{\mu_3} \in M_2(\mathbb{C})$$

with corresponding rank-two projections $\rho(p_\mu)$, $\mu \in \widehat{H}$.

The representation theory of G , the stabilizer of the remaining corners $(0, 0)$ and $(\frac{1}{2}, \frac{1}{2})$ of Z , is as follows: there is the ‘standard’ representation $\lambda: G \rightarrow O(2, \mathbb{R}) \subset U(2)$ (this is irreducible). The other irreducible representations are one-dimensional and correspond to the representations of the quotient group $G/\langle R^2 \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. They are listed as $\{1_G, \chi_1, \chi_2, \chi_3\}$ with

$$\chi_1(R) = -\chi_1(S) = 1, \quad \chi_2(R) = \chi_2(S) = -1 \quad \text{and} \quad \chi_3(R) = -\chi_3(S) = -1.$$

Therefore the set of irreducible representations of G is $\{1, \chi_1, \chi_2, \chi_3, \lambda\}$. Representing $C^*(G) \cong \mathcal{K}(\ell^2(G))^G$ as a subalgebra of $\mathcal{K}(\ell^2(G)) \cong M_8(\mathbb{C})$ gives one block

$M_2(\mathbb{C})$ with multiplicity 2 and four one-dimensional blocks. With respect to a suitable chosen base of $\ell^2(G) \cong \mathbb{C}^8$, we obtain a representation as matrices of the form

$$\begin{pmatrix} C_\lambda & & & & & & & \\ & C_\lambda & & & & & & \\ & & d_1 & & & & & \\ & & & d_{\chi_1} & & & & \\ & & & & d_{\chi_2} & & & \\ & & & & & d_{\chi_3} & & \\ & & & & & & & \end{pmatrix} \quad C_\lambda \in M_2(\mathbb{C}), d_1, d_{\chi_1}, d_{\chi_2}, d_{\chi_3} \in \mathbb{C},$$

where the lower diagonal entries act on the images of the projections $\rho(p_\chi)$ for $\chi \in \{1_G, \chi_1, \chi_2, \chi_3\}$ and the block $\begin{pmatrix} C_\lambda & \\ & C_\lambda \end{pmatrix}$ acts on the four-dimensional space $\rho(p_\lambda)\ell^2(G)$.

To understand the structure of the algebra

$$C(\mathbb{T}^2) \rtimes G \cong \{f \in C(Z, \mathcal{K}(\ell^2(G))) : f(s, t) \in \mathcal{K}(\ell^2(G))^{G_{(s,t)}}\},$$

we need to understand what happens at the three corners $(0, 0)$, $(\frac{1}{2}, \frac{1}{2})$, $(0, \frac{1}{2})$ of the fundamental domain Z when approached on the border lines of Z . So assume that $f \in C(\mathbb{T}^2) \rtimes G$ is represented as a function $f : Z \rightarrow M_8(\mathbb{C}) \cong \mathcal{K}(\ell^2(G))$. Since the stabilizers of the corners contain the stabilizers of the adjacent border lines, we see from the above discussion that the fibers $\mathcal{K}(\ell^2(G))^{G_{(s,t)}}$ at the corners must be contained in the intersections of the fibers at the adjacent border lines. Proposition 3.4 above tells us, how these inclusions look like: Let us consider the corner $(0, 0)$. The adjacent border lines have stabilizers $K_1 = G_{(s,s)} = \langle RS \rangle$ and $K_2 = G_{(0,t)} = \langle R^2 S \rangle$ respectively. A short computation shows that the restriction of λ to K_1 and K_2 decomposes into the direct sum $1_{K_i} \oplus \epsilon_{K_i}$ for $i = 1, 2$. So each of the two 4×4 -blocks in the decomposition

$$\mathcal{K}(\ell^2(G))^{K_i} = \left\{ \begin{pmatrix} A_{1_{K_i}} & 0 \\ 0 & A_{\epsilon_{K_i}} \end{pmatrix} : A_{1_{K_i}}, A_{\epsilon_{K_i}} \in M_4(\mathbb{C}) \right\},$$

contains exactly one copy of the two by two blocks C_λ in the decomposition of $\mathcal{K}(\ell^2(G))^G$. Consider now the one-dimensional representations $1_G, \chi_1, \chi_2, \chi_3$ of G . If we restrict these representations to K_1 we see that 1_G and χ_2 restrict to trivial character 1_{K_1} and χ_1, χ_3 restrict to the non-trivial character ϵ_{K_1} . Thus, the 4×4 block $A_{1_{K_1}}$ contains the diagonal entries d_1, d_{χ_2} and the block $A_{\epsilon_{K_1}}$ contains the diagonal entries d_{χ_1}, d_{χ_3} .

On the other side, if we restrict $1_G, \chi_1, \chi_2, \chi_3$ to the subgroup $K_2 = G_{(0,t)}$ we see that $1_G, \chi_3$ restrict to 1_{K_2} and χ_1, χ_2 restrict to the non-trivial character ϵ_{K_2} . We therefore see that, different from the case $K_1 = G_{(s,s)}$, $A_{1_{K_2}}$ contains the diagonal entries corresponding to the characters d_1, d_{χ_3} and the block $A_{\epsilon_{K_2}}$ contains the diagonal entries corresponding to d_{χ_1}, d_{χ_2} . So, even in this simple example, we get a quite intricate structure of the algebra $C(\mathbb{T}^2) \rtimes G$ at the fibers with nontrivial stabilizers. We shall revisit this example in the following section.

We now proceed with the general theory:

Definition 3.6. Let X be a proper G -space. We define the *stabilizer group bundle* $\text{Stab}(X)$ as

$$\text{Stab}(X) = \{(x, g) : x \in X, g \in G_x\}.$$

and we define

$$\text{Stab}(X)^\wedge := \{(x, \sigma) : x \in X, \sigma \in \widehat{G}_x\}.$$

Note that G acts on $\text{Stab}(X)^\wedge$ by

$$g(x, \sigma) = (gx, g\sigma) \quad \text{with} \quad g\sigma := \sigma \circ C_g^{-1},$$

where $C_g : G_x \rightarrow G_{gx}$ is the isomorphism given by conjugation with g . For each $(x, \sigma) \in \text{Stab}(X)^\wedge$ consider the induced representation $\pi_x^\sigma = P_x^\sigma \times U_x^\sigma$ acting on the Hilbert space \mathcal{H}_{U^σ} as defined in (3.1) and (3.2). One easily checks that π_x^σ is unitarily equivalent to $\pi_{gx}^{g\sigma}$, the equivalence being given by the unitary

$$W : \mathcal{H}_\sigma \rightarrow \mathcal{H}_{\sigma \circ C_g^{-1}}; (W\xi)(t) = \sqrt{\Delta(g)}\xi(tg).$$

In what follows, we shall also write $\text{Ind}_{G_x}^G(x, \sigma)$ for the representation π_x^σ , indicating that it is the induced representation in the classical sense of Mackey, Glimm and others. Thus, as a corollary of the above lemma, we obtain a proof of the following theorem, which is a well-known special case of the general Mackey-Green-Rieffel machine for crossed products.

Theorem 3.7 (Mackey-Rieffel-Green). *The map*

$$\text{Ind} : \text{Stab}(X)^\wedge \rightarrow (C_0(X) \rtimes G)^\wedge; (x, \sigma) \mapsto \text{Ind}_{G_x}^G(x, \sigma) = \pi_x^\sigma$$

factors through a set bijection (which we also denote Ind) between the orbit space $G \backslash \text{Stab}(X)^\wedge$ and the space $(C_0(X) \rtimes G)^\wedge$ of equivalence classes of irreducible $$ -representations of $C_0(X) \rtimes G$.*

Proof. The proof is an easy combination of the bundle structure of $C_0(X) \rtimes G \cong C_0(X \times_{G, \text{Ad } \rho} \mathcal{K})$ and the description of the fibre \mathcal{K}^{G_x} as given in the previous lemma. \square

If we look at the trivial representation $1_{G_x} : G_x \rightarrow \{1\}$, it follows from Lemma 3.1 that the corresponding summand of the fibre \mathcal{K}^{G_x} of $C_0(X) \rtimes G$ is given by $\mathcal{K}(L^2(G)_{1_{G_x}})$ with

$$(3.5) \quad L^2(G)_{1_{G_x}} = \{\xi \in L^2(G) : \xi(gk) = \xi(g) \forall k \in G_x\} \cong L^2(G/G_x).$$

Let $c : X \rightarrow [0, 1]$ be a *cut-off function* for the proper G -space X , which means that c is a continuous function with compact support on any G -compact subset of X (a closed subset Y of X is called *G -compact* if it is G -invariant with $G \backslash Y$ compact) such that $\int_G c(g^{-1}x)^2 dg = 1$ for all $x \in X$. Then

$$(3.6) \quad p_X(g, x) = \sqrt{\Delta(g^{-1})}c(g^{-1}x)c(x)$$

determines a projection $p_X \in M(C_0(X) \rtimes G)$ via convolution given by the same formula as for the multiplication on $C_c(G \times X) \subset C_0(X) \rtimes G$. Note that $p_X \in C_0(X) \rtimes G$ if and only if X is G -compact, while in general, $f \cdot p_X \in C_0(X) \rtimes G$ for every $f \in C_0(G \backslash X) \subseteq ZM(C_0(X) \rtimes G)$.

Remark 3.8. We make the following remarks about cut-off functions and their associated projections.

- If X is compact G must be compact too, and in this case with respect to normalised Haar measure on G , the constant function $c(x) := 1$ for all $x \in X$ is a cut-off function. In this case $C^*(G)$ is a subalgebra of $C(X) \rtimes G$, and the projection p_X is in the subalgebra: it is the constant function 1 on G and projects to the space of G -fixed vectors in any representation of G .
- For any proper action, the space of square roots of cut-off functions is convex, so the space of cut-off functions is contractible and hence any two projections p_X are homotopic.

Lemma 3.9. *Let $c : X \rightarrow [0, 1]$ and $p_X \in M(C_0(X) \rtimes G)$ be as above and let $\Phi : C_0(X) \rtimes G \xrightarrow{\cong} C_0(X \times_G \mathcal{K})$ denote the isomorphism of Corollary 2.11. For each $x \in X$ let c_x denote the unit vector in $L^2(G)$ given by $c_x(g) = \sqrt{\Delta(g^{-1})}c(gx)$ and let $p_x \in \mathcal{K}(L^2(G))$ denote the image of p_X under the evaluation map $q_x : C_0(X) \rtimes G \rightarrow \mathcal{K}^{G_x}$ as described in Remark 2.12. Then $c_x \in L^2(G)_{1_{G_x}}$ and p_x is the orthogonal projection onto $\mathbb{C} \cdot c_x$.*

Proof. The first assertion follows from the definition of c_x together with the fact that the modular function vanishes on compact subgroups of G . For the second assertion observe that it follows from Remark 2.12 that p_x acts on $L^2(G)$ via convolution with the function $p_x \in C_c(G \times G/G_x) \subseteq C_0(G/G_x) \rtimes G$ given by

$$p_x(g, hG_x) = \Delta(g^{-1}h)c_x(g^{-1}h)c_x(h).$$

If $\xi \in L^2(G)$ is arbitrary, we get

$$\begin{aligned} (p_x \xi)(h) &= \int_G p_x(g, hG_x) \xi(g^{-1}h) dg \\ &= \int_G \Delta(g^{-1}h)c_x(g^{-1}h)c_x(h)\xi(g^{-1}h) dg \\ &= \left(\int_G c_x(g)\xi(g) dg \right) c_x(h) \\ &= (\langle \xi, c_x \rangle c_x)(h), \end{aligned}$$

where the second to last equation follows from the transformation $g^{-1}h \mapsto g$. \square

It follows from the above lemma that the projection $p_X \in M(C_0(X) \rtimes G)$ constructed above is a continuous field of rank-one projections on $G \backslash X$ such that under the decomposition of each fibre $\mathcal{K}^{G_x} \cong \bigoplus_{\sigma \in \widehat{G_x}} \mathcal{K}(\mathcal{H}_\sigma)$ as in part (iii) of Lemma 3.1, the restriction of p_X to that fibre lies in the component $\mathcal{K}(\mathcal{H}_{1_{G_x}})$. It follows in particular that $C_0(G \backslash X)$ is isomorphic to the corner $p_X(C_0(X) \rtimes G)p_X$ via $f \mapsto f \cdot p_X$, and thus $C_0(G \backslash X) \cong p_X(C_0(X) \rtimes G)p_X$ is Morita equivalent to the ideal $I_X = (C_0(X) \rtimes G)p_X(C_0(X) \rtimes G)$ of $C_0(X) \rtimes G$ generated by p_X (this is a general fact about corners.)

Lemma 3.1 implies that under the isomorphism $C_0(X) \rtimes G \cong C_0(X \times_G \mathcal{K})$ we get

$$(3.7) \quad I_X = \{F \in C_0(X \times_G \mathcal{K}) : F(x) \in \mathcal{K}(L^2(G)_{1_{G_x}}) \forall x \in X\}.$$

It follows in particular that the ideal I_X does not depend on the particular choice of the cut-off function $c : X \rightarrow [0, 1]$ and the corresponding projection p_X . If the action of G on X is *free and proper*, then it is immediate from (3.7) and the description of $C_0(X) \rtimes G$ in Corollary 2.11 and the following Remark 2.12, that $I_X = C_0(X) \rtimes G$. We thus recover the well-known theorem, due to Phillip Green (see [29]) that $C_0(G \backslash X) \sim_M C_0(X) \rtimes G$ for a free and proper action of a locally compact group G .

Example 3.10. The above can be made rather explicit in the case of finite group actions. For definiteness, we let $G = \mathbb{Z}/2$, X is compact. By Remark 3.8, we may take $p_X \in C(X) \rtimes G$ to be $p_X(x, g) = \frac{1}{|G|}$, while $C(X) \rtimes G \cong C(X, \mathcal{K}(\ell^2(G)))^G$. We can consider $\mathcal{K}(\ell^2(G))$ as 2×2 -matrices, and the G -invariance says the elements in $C(X, \mathcal{K}(\ell^2(G)))^G$ must have the form

$$a = \begin{bmatrix} f & g \\ \sigma(g) & \sigma(f) \end{bmatrix}$$

for some $f, g \in C(X)$, where σ is the underlying order two automorphism of $C(X)$. The projection p_X corresponds to the matrix $\frac{1}{|G|} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. The ideal $I_X = (C(X) \rtimes G)p_X(C(X) \rtimes G)$ is then given by the closed linear hull of matrices of the form $\begin{bmatrix} fg & f\sigma(g) \\ \sigma(f)g & \sigma(fg) \end{bmatrix}$, $f, g \in C(X)$, while the corner $p_X(C(X) \rtimes G)p_X$ consists of all matrices of the form $\begin{bmatrix} f & f \\ f & f \end{bmatrix}$ where $\sigma(f) = f$. This C^* -algebra is isomorphic to $C(G \setminus X)$.

It is also useful to give a representation theoretic description of the ideal I_X in terms of the Mackey-Rieffel-Green machine of Theorem 3.7. For this recall that for any closed two-sided ideal I in a C^* -algebra A the spectrum \widehat{I} includes as an open subset of \widehat{A} via (unique) extension of irreducible representations from I to A , and the resulting correspondence $I \subseteq A \leftrightarrow \widehat{I} \subseteq \widehat{A}$ is one-to-one. Thus the ideal I_X in $C_0(X) \rtimes G$ is uniquely determined by the set of irreducible representations of $C_0(X) \rtimes G$ which do not vanish on I_X . The above results now combine to the following:

Proposition 3.11. *The irreducible representations of $C_0(X) \rtimes G$ which correspond to the ideal I_X are precisely the representations of the form $\pi_x^{1_{G_x}} = \text{Ind}_{G_x}^G(x, 1_{G_x})$, $x \in X$, where 1_{G_x} denotes the trivial representation of the stabilizer G_x .*

Example 3.12. The above representation theoretic description of the Ideal I_X makes it easy to identify this ideal in the case of the crossed product $C(\mathbb{T}^2) \rtimes G$ with $G = D_4$ acting on $X = \mathbb{T}^2$ as described in Examples 2.6 and 3.5. If we realize

$$C(\mathbb{T}^2) \rtimes G = \{f \in C(Z, \mathcal{K}(\ell^2(G))) : f(s, t) \in \mathcal{K}(\ell^2(G))^{G_{(s,t)}}\}$$

as in Example 3.5, then, with respect to the description of the fibers $\mathcal{K}(\ell^2(G))^{G_{(s,t)}}$ as given in that example, the ideal $I_{\mathbb{T}^2}$ consists of those functions which take arbitrary values in the interior of Z and which take values in the corners A_1 at the boundaries of Z and in d_1 and B_1 at the corners $(0, 0)$, $(\frac{1}{2}, \frac{1}{2})$ and $(0, \frac{1}{2})$, respectively.

Recall from §1 that every proper G -space X is locally induced by compact subgroups of G , which means that each $x \in X$ has a G -invariant open neighborhood U of the form $U \cong G \times_K Y$. For later use we need to compare the Mackey-Rieffel-Green map of $C_0(G \times_K Y) \rtimes G$ with that of $C_0(Y) \rtimes K$. By a version of Green's imprimitivity theorem we know that $C_0(G \times_K Y) \rtimes G$ is Morita equivalent to $C_0(Y) \rtimes K$. The imprimitivity bimodule E is given by a completion of $E_0 = C_c(G \times Y)$ with underlying pre-Hilbert $C_c(K \times Y)$ -structure given by

$$(3.8) \quad \begin{aligned} \langle \xi, \eta \rangle_{C_c(K \times Y)}(k, y) &= \int_G \overline{\xi(g^{-1}, y)} \eta(g^{-1}k, k^{-1}y) dg \\ \xi \cdot b(g, y) &= \int_K \xi(gk^{-1}, ky) b(k, ky) dk \end{aligned}$$

for $\xi, \eta \in E_0$ and $b \in C_c(K \times Y) \subseteq C_0(Y) \rtimes K$, and with left action of the dense subalgebra $C_c(G \times (G \times_K Y))$ of $C_0(G \times_K Y) \rtimes G$ on E_0 given by the covariant representation (P, L) such that

$$(3.9) \quad (P(F)\xi)(g, y) = F([g, y])\xi(g, y) \quad \text{and} \quad (L(t)\xi)(g, y) = \Delta(t)^{1/2}\xi(t^{-1}g, y),$$

for $F \in C_0(G \times_K Y)$. These formulas follow from [48, Corollary 4.17] by identifying $C_0(G \times_K Y)$ with $C_0(G \times_K C_0(Y))$ ($= \text{Ind}_K^G C_0(Y)$ in the notation of [48]). Induction of representations from $C(Y) \rtimes K$ to $C_0(G \times_K Y) \rtimes G$ via the imprimitivity bimodule E induces a homeomorphism between $(C_0(Y) \rtimes K)^\wedge$ and $(C_0(G \times_K Y) \rtimes G)^\wedge$.

Proposition 3.13. *Suppose that K is a compact subgroup of G , Y is a K -space and $X = G \times_K Y$. Then there is a commutative diagram of bijective maps*

$$\begin{array}{ccc} K \backslash \text{Stab}(Y)^\wedge & \xrightarrow{\iota} & G \backslash \text{Stab}(X)^\wedge \\ \text{Ind}^K \downarrow & & \downarrow \text{Ind}^G \\ (C_0(Y) \rtimes K)^\wedge & \xrightarrow{\text{Ind}^E} & (C_0(X) \rtimes G)^\wedge \end{array}$$

where the vertical maps are the respective induction maps of Theorem 3.7, the lower horizontal map is induction via the imprimitivity bimodule constructed above and the upper horizontal map is given by the map on orbit spaces induced by the inclusion $\iota : \text{Stab}(Y)^\wedge \rightarrow \text{Stab}(X)^\wedge$, $(y, \sigma) \mapsto ([e, y], \sigma)$.

Proof. Let $(y, \sigma) \in \text{Stab}(Y)^\wedge$. Let τ_y^σ denote the representation of $C_0(Y) \rtimes K$ induced from (y, σ) and let π_y^σ denote the representation of $C_0(G \times_K Y) \rtimes G$ induced from $([e, y], \sigma)$. Let \mathcal{H}_σ^K and \mathcal{H}_σ^G denote the respective Hilbert spaces on which they act. We have to check that $\text{Ind}^E \tau_y^\sigma \cong \pi_y^\sigma$. Recall that $\text{Ind}^E \tau_y^\sigma$ acts on the Hilbert space $E \otimes_{C_0(Y) \rtimes K} \mathcal{H}_\sigma^K$ via the left action of $C_0(G \times_K Y) \rtimes G$ on E , as specified in (3.9). Recall from (3.1) that $\mathcal{H}_\sigma^K = \{\varphi \in L^2(K, V_\sigma) : \varphi(kl) = \sigma(l^{-1})\varphi(k) \forall l \in K_y\}$ (since K is unimodular) and similarly for \mathcal{H}_σ^G . We claim that there is a unique unitary operator

$$\Phi : E \otimes_{C_0(Y) \rtimes K} \mathcal{H}_\sigma^K \rightarrow \mathcal{H}_\sigma^G$$

given on elementary tensors $\xi \otimes \varphi$, $\xi \in E_0$, $\varphi \in \mathcal{H}_\sigma^K$, by

$$\Phi(\xi \otimes \varphi)(g) = \Delta(g)^{-1/2} \int_K \xi(gk^{-1}, ky) \varphi(k) dk.$$

A quick computation shows that $\Phi(\xi \otimes \varphi)(gl) = \Delta_G(l)^{-1/2} \sigma(l^{-1}) \Phi(\xi \otimes \varphi)(g)$. To see that Φ preserves the inner products we compute for all $\xi, \eta \in E_0$ and $\varphi, \psi \in \mathcal{H}_\sigma^K$:

$$\begin{aligned} \langle \Phi(\xi \otimes \varphi), \Phi(\eta \otimes \psi) \rangle_{\mathcal{H}_\sigma^G} &= \int_G \langle \Phi(\xi \otimes \varphi)(g), \Phi(\eta \otimes \psi)(g) \rangle_{V_\sigma} dg \\ &= \int_G \Delta(g^{-1}) \int_K \int_K \langle \xi(gk^{-1}, ky) \varphi(k), \eta(gl^{-1}, ly) \psi(l) \rangle_{V_\sigma} dl dk dg \\ &= \int_G \int_K \int_K \overline{\xi(g^{-1}k^{-1}, ky)} \eta(g^{-1}l^{-1}, ly) \langle \varphi(k), \psi(l) \rangle_{V_\sigma} dl dk dg \end{aligned}$$

while on the other side we get

$$\langle \xi \otimes \varphi, \eta \otimes \psi \rangle_{E \otimes_{C_0(Y) \rtimes K} \mathcal{H}_\sigma^K} = \langle \tau_y^\sigma(\langle \eta, \xi \rangle_E) \varphi, \psi \rangle_{\mathcal{H}_\sigma^K}.$$

For a function $f \in C_c(K \times Y) \subseteq C_0(Y) \rtimes K$ the operator $\tau_y^\sigma(f)$ acts on $\varphi \in \mathcal{H}_\sigma^K$ by

$$(\tau_y^\sigma(f)\varphi)(l) = \int_K f(k, ly) \varphi(k^{-1}l) dk.$$

Applying this together with the formula for $\langle \xi, \eta \rangle_E$ as given in (3.8), we get

$$\begin{aligned} \langle \xi \otimes \varphi, \eta \otimes \psi \rangle_{E \otimes_{C_0(Y) \rtimes K} \mathcal{H}_\sigma^K} &= \int_K \int_K \langle \langle \eta, \xi \rangle_E(l, ky) \varphi(l^{-1}k), \psi(k) \rangle_{V_\sigma} dl dk \\ &= \int_K \int_K \int_G \overline{\xi(g^{-1}k, k^{-1}ly)} \eta(g^{-1}, ly) \langle \varphi(k^{-1}l), \psi(l) \rangle_{V_\sigma} dg dl dk \\ &= \int_G \int_K \int_K \overline{\xi(g^{-1}k^{-1}, ky)} \eta(g^{-1}l^{-1}, ly) \langle \varphi(k), \psi(l) \rangle_{V_\sigma} dl dk dg \\ &= \langle \Phi(\xi \otimes \varphi), \Phi(\eta \otimes \psi) \rangle_{\mathcal{H}_\sigma^G}, \end{aligned}$$

where the second to last equation follows from Fubini and the transformation $g \mapsto lg$ followed by the transformation $k \mapsto lk^{-1}$.

It follows now that Φ extends to a well defined isometry from $E \otimes_{C_0(Y) \rtimes K} \mathcal{H}_\sigma^K$ into \mathcal{H}_σ^G . We now show that it intertwines the representations $\text{Ind}^E \tau_y^\sigma$ and π_y^σ . Since these representations are irreducible, this will then also imply surjectivity of Φ . For the left action of $F \in C_0(G \times_K Y)$ we check

$$\begin{aligned} \Phi(P(F)\xi \otimes \varphi)(g) &= \Delta(g)^{-1/2} \int_K F([gk^{-1}, ky])\xi(gk^{-1}, ky)\varphi(k) dk \\ &= F([g, y])\Phi(\xi \otimes \varphi)(g) = (P_{[e, y]}^\sigma(F)\Phi(\xi \otimes \varphi))(g), \end{aligned}$$

where we used the equations $[gk^{-1}, ky] = [g, y] = g \cdot [e, y]$. Similarly, for the actions of G we easily check

$$\Phi(L(t)\xi \otimes \varphi)(g) = \Phi(\xi \otimes \varphi)(t^{-1}g) = (U_y^\sigma(t)\Phi(\xi \otimes \varphi))(g),$$

which now completes the proof. \square

Recall that any A – B -imprimitivity bimodule E induces a bijection of ideals in A and B . Under the correspondence between ideals in A (resp. B) and open subsets of \widehat{A} (resp. \widehat{B}) the correspondence of ideals of A and B induced by E is compatible with the correspondence of open subsets in \widehat{A} and \widehat{B} given by the homeomorphism $\text{ind}^E : \widehat{B} \rightarrow \widehat{A}$. This all follows from the Rieffel-correspondence as explained in [42, Chapter 3.3]. Using these facts together with the above Proposition 3.11 and Proposition 3.13 we get

Corollary 3.14. *Suppose that K is a compact subgroup of G , Y is a K -space and $X = G \times_K Y$. Then under the above described Morita equivalence between $C_0(X) \rtimes G$ and $C_0(Y) \rtimes K$ the ideal I_X in $C_0(X) \rtimes G$ corresponds to the ideal I_Y in $C_0(Y) \rtimes K$.*

Remark 3.15. Analogues of Proposition 3.13 and Corollary 3.14 are also true if the compact subgroup K is replaced by any given closed subgroup H of G which acts properly on the space Y . The arguments are exactly the same—only some formulas become a bit more complicated due to the appearance of the modular function on H . Since we only need the compact case below, we restricted to this case here.

4. THE SPECTRUM OF $C_0(X) \rtimes G$

As first step to obtain any further progress for K -theory computations of crossed products by proper actions with non-isolated free orbits, it should be useful to obtain a better understanding of the ideal structure of the crossed products. Since closed ideals in $C_0(X) \rtimes G$ correspond to open subsets of $(C_0(X) \rtimes G)^\wedge$, this problem is strongly related to a computation of the topology of the representation space $(C_0(X) \rtimes G)^\wedge$.

So in this section we give a detailed description of the topology of $(C_0(X) \rtimes G)^\wedge$ in terms of the bijection with the parameter space $G \backslash \text{Stab}(X)^\wedge$ as in Theorem 3.7. To be more precise, we shall introduce a topology on $\text{Stab}(X)^\wedge$ such that the bijection of Theorem 3.7 becomes a homeomorphism, if $G \backslash \text{Stab}(X)^\wedge$ carries the corresponding quotient topology. Note that Baggett gives in [6] a general description of the topology of the unitary duals of semi-direct product groups $N \rtimes K$, with N abelian and K compact in terms of the Fell-topology on the set of subgroup representations of K . (This is the space of all pairs (L, τ) , where L is a subgroup of K and τ is a unitary representation of L , see [28, 32] for the definition.) Since $(N \rtimes K)^\wedge = C^*(N \rtimes K)^\wedge \cong (C_0(\widehat{N}) \rtimes K)^\wedge$ the study of such semi-direct products can be regarded as a special case of the study of crossed products by proper actions.

Since Fell's topology on subgroup representations is not very easy to understand, we aim to define a suitable topology on $\text{Stab}(X)^\wedge$ without using this construction. To make this possible, we restrict our attention to actions which satisfy Palais's slice property (SP). Recall that this means that the proper G -space X is locally induced from actions of the stabilizers. Recall also that by Palais's Theorem (see Theorem 1.6), property (SP) is always satisfied if G is a Lie-group.

Let us introduce some further notation: If X is a proper G -space with property (SP), then for every $x \in X$ we define

$$S_x := \{y \in X : G_y \subseteq G_x\}.$$

By an *almost slice* at $x \in X$ we shall understand any set of the form $W \cdot V_x$, where W is an open neighborhood of e in G and V_x is an open neighborhood of x in S_x , i.e., $V_x = S_x \cap U_x$ for some open neighborhood U_x of x in X . We denote by \mathcal{AS}_x the set of all almost slices at x .

Lemma 4.1. *Let X be a proper G -space with property (SP). Then the set \mathcal{AS}_x of all almost slices at x forms an open neighborhood base at x .*

Proof. Let WV_x be any almost slice at x . To see that it is open in X let $y \in WV_x$ be arbitrary. Let $g \in W$ such that $y \in gV_x$. Let Y_y be a local slice at y , i.e., there is an open neighborhood U_y of y such that $U_y = G \cdot Y_y \cong G \times_{G_y} Y_y$. Then $Y_y \subseteq S_y \subseteq S_{gx}$. Thus, passing to a smaller local slice if necessary, we may assume that $Y_y \subseteq gV_x$. Now choose an open neighborhood W' of e in G such that $W'g \subseteq W$. Then $W'Y_y \subseteq W'gV_x \subseteq WV_x$ is an open neighborhood of y contained in WV_x . (Note that the map $G \times Y_y \rightarrow G \cdot Y_y$ is open since it coincides with the quotient map $G \times Y_y \rightarrow G_y \backslash (G \times Y_y) = G \times_{G_y} Y_y$.)

Conversely, it follows from the continuity of the action that every open neighborhood V of x in X contains an almost slice WV_x at x , which finishes the proof. \square

In what follows, if τ and σ are representations, we write $\tau \leq \sigma$ if τ is a subrepresentation of σ .

Definition 4.2. Suppose that X is a proper G -space which satisfies (SP). If $(x, \sigma) \in \text{Stab}(X)^\wedge$ and if $WV_x \in \mathcal{AS}_x$ we say that a pair $(z, \tau) \in \text{Stab}(X)^\wedge$ lies in the set $U(x, \sigma, WV_x)$ if and only if there exists $y \in V_x$ and $g \in W$ such that $z = gy$ and $\tau \leq g\sigma|_{G_z}$. Alternatively one could describe the sets $U(x, \sigma, WV_x)$ as the product $W \cdot U(x, \sigma, V_x)$ with

$$U(x, \sigma, V_x) := \{(y, \tau) \in \text{Stab}(X)^\wedge : y \in V_x, \tau \leq \sigma|_{G_y}\}.$$

We further define

$$\mathcal{U}_{(x, \sigma)} := \{U(x, \sigma, WV_x) : WV_x \in \mathcal{AS}_x\}.$$

Lemma 4.3. *There is a topology on $\text{Stab}(X)^\wedge$ such that the elements of $\mathcal{U}_{(x, \sigma)}$ form a base of open neighborhoods for the element $(x, \sigma) \in \text{Stab}(X)^\wedge$ in this topology. Moreover, the canonical action of G on $\text{Stab}(X)^\wedge$ is continuous with respect to this topology.*

Proof. We have to show that if (x, σ) lies in the intersection of two sets $U(x_1, \sigma_1, W_1V_1)$ and $U(x_2, \sigma_2, W_2V_2)$, then there exists an almost slice WV at x such that

$$U(x, \sigma, WV) \subseteq U(x_1, \sigma_1, W_1V_1) \cap U(x_2, \sigma_2, W_2V_2).$$

If this is shown, then the union $\bigcup_{(x, \sigma) \in \text{Stab}(X)^\wedge} \mathcal{U}_{(x, \sigma)}$ forms a base of a topology with the required properties.

For this let $g_1 \in W_1$ and $g_2 \in W_2$ such that $y_i := g_i^{-1}x \in V_i$ and such that $g_i^{-1}\sigma$ is a sub-representation of $\sigma_i|_{G_{y_i}}$ for $i = 1, 2$. Since $g_iS_x = S_{g_ix} \subseteq S_{x_i}$ for $i = 1, 2$, we

may choose an open neighborhood V of x in S_x such that $g_i V \subseteq V_i$ for $i = 1, 2$. We may then also find a symmetric open neighborhood W of e in G such that $g_i W \subseteq W_i$ and $WV \subseteq W_i V_i$ for $i = 1, 2$. We want to show that $U(x, \sigma, WV) \subseteq U(x_i, \sigma_i, W_i V_i)$ for $i = 1, 2$. Since $\mathcal{U}_{(x, \sigma)}$ is closed under finite intersections, which follows easily from the definitions, it is enough to show this for $i = 1$.

So let $(y, \tau) \in U(x, \sigma, WV)$. Let $g \in W$ such that $gy \in V$ and $g\tau$ is a sub-representation of $\sigma|_{G_{gy}}$. Then $g_1 g \in W_1$ with $g_1 gy \in V_1$ and $g_1 g\tau$ is a sub-representation of $g_1(\sigma|_{G_{gy}})$, which is a subrepresentation of $(\sigma_1|_{G_{g_1 y}})|_{G_{g_1 gy}} = \sigma_1|_{G_{g_1 gy}}$. Thus $(y, \tau) \in U(x_1, \sigma_1, W_1 V_1)$.

To see that the action of G on $\text{Stab}(X)^\wedge$ is continuous let $U(gx, g\sigma, WV_{gx})$ be a given neighborhood of $(gx, g\sigma)$. Then, if W_0 is an open neighborhood of e in G such that $gW_0^2 g^{-1} \subseteq W$ and if we define $V_x := g^{-1} \cdot V_{gx}$, we get $(gW_0) \cdot U(x, \sigma, W_0 V_x) = (gW_0^2 g^{-1}) \cdot U(gx, g\sigma, V_{gx}) \subseteq U(gx, g\sigma, WV_{gx})$ and we are done. \square

Remark 4.4. In the case where G is a discrete group, the description of the topology on $\text{Stab}(X)^\wedge$ becomes easier to describe since for every point $x \in X$ the set $S_x \subseteq X$ is open in X , and hence contains a neighborhood base of sets V_x . (Observe also that $V_x = WV_x$ for $W = \{e\}$.) Therefore a neighborhood base of a pair $(x, \sigma) \in \text{Stab}(X)^\wedge$ is given by the sets $U(x, \sigma, V_x)$, where V_x runs through all open neighborhoods of x in S_x . In this case, we may even replace the V_x by local slices Y_x , since for discrete G the local slices are also open in X .

Remark 4.5. Another obvious approach to define basic neighborhoods for the topology of $\text{Stab}(X)^\wedge$ would be to consider the sets $U(x, \sigma, WY_x) := W \cdot U(x, \sigma, Y_x)$ where the Y_x are local slices at x and $U(x, \sigma, Y_x) := \{(y, \tau) : y \in Y_x, \tau \leq \sigma|_{G_y}\}$. We actually believe that these sets do form a neighborhood base of the above defined topology, but we lack a proof. In particular, it is not clear to us whether the intersection of two sets of this form will contain a third one of this form. The difficulty comes from the fact that the intersection of two local slices at x might not be a local slice at x – in fact the intersection will very often only contain the point x . However, it follows from the above remark that these problems disappear if G is discrete.

In what follows, we always equip $(C_0(X) \rtimes G)^\wedge$ with the Jacobson topology and $G \backslash \text{Stab}(X)^\wedge$ with the quotient topology of the above defined topology on $\text{Stab}(X)^\wedge$.

Theorem 4.6. *Assume that X is a proper G -space which satisfies Palais's slice property (SP). Let $\text{Stab}(X)^\wedge$ be equipped with the topology constructed above. Then the bijection $[(x, \sigma)] \mapsto \pi_x^\sigma$ between $G \backslash \text{Stab}(X)^\wedge$ and $(C_0(X) \rtimes G)^\wedge$ of Theorem 3.7 is a homeomorphism.*

Before we start with the proof, we have to recall the definition of the *Fell-topology* on the set $\text{Rep}(A)$ of all equivalence classes of representations of a C^* -algebra A with dimension dominated by some fixed cardinal κ (κ is always chosen so big, that all representations we care for lie in $\text{Rep}(A)$). A neighborhood base for the Fell topology is given by the collection of all sets of the form

$$U(I_1, \dots, I_l) = \{\pi \in \text{Rep}(A) : \pi(I_i) \neq \{0\} \text{ for all } 1 \leq i \leq l\},$$

where I_1, \dots, I_l is any given finite collection of closed two-sided ideals in A . If we restrict this topology to the set \hat{A} of equivalence classes of irreducible representations of A , we recover the usual Jacobson topology on \hat{A} .

Convergence of nets in $\text{Rep}(A)$ can conveniently be described in terms of *weak containment*: If $\pi \in \text{Rep}(A)$ and R is a subset of $\text{Rep}(A)$, then π is said to be *weakly contained* in R (denoted $\pi \prec R$) if $\ker \pi \supseteq \bigcap \{\ker \rho : \rho \in R\}$. Two subsets S, R of $\text{Rep}(A)$ are said to be *weakly equivalent* ($S \sim R$) if $\sigma \prec R$ for all $\sigma \in S$ and $\rho \prec S$ for all $\rho \in R$.

Lemma 4.7 (Fell). *Let $(\pi_j)_{j \in J}$ be a net in $\text{Rep}(A)$ and let $\pi, \rho \in \text{Rep}(A)$. Then*

- (i) $\pi_j \rightarrow \pi$ if and only if π is weakly contained in every subnet of $(\pi_j)_{j \in J}$.
- (ii) If $\pi_j \rightarrow \pi$ and if $\rho \prec \pi$, then $\pi_j \rightarrow \rho$.

For the proof see [32, Propositions 1.2 and 1.3]. We should also note that by construction of the Fell-topology, the topology can only see the kernel of a representation and not the representation itself – that means in particular that if we replace a net (π_i) by some other net $(\tilde{\pi}_i)$ with $\ker \tilde{\pi}_i = \ker \pi_i$ for all $i \in I$, then both nets have the same limit sets!

Suppose now that A, B are two C^* -algebras and let ${}_A E_B$ be a Hilbert $A - B$ -bimodule. By this we understand a Hilbert B -module E_B together with a fixed $*$ -homomorphism $\Phi : A \rightarrow \mathcal{L}_B(E)$. Then E induces an induction map (due to Rieffel)

$$\text{Ind}^E : \text{Rep}(B) \rightarrow \text{Rep}(A)$$

which sends a representation $\pi \in \text{Rep}(B)$ to the induced representation $\text{Ind}^\pi \in \text{Rep}(A)$, which acts on the balanced tensor product $E \otimes_B \mathcal{H}_\pi$ via

$$\text{Ind}^E \pi(a)(\xi \otimes v) = \Phi(a)\xi \otimes 1.$$

One can check that

$$\ker(\text{Ind}^E \pi) = \{a \in A : \Phi(a)E \subseteq E \cdot (\ker \pi)\}$$

from which it follows that induction via ${}_A E_B$ preserves weak containment, and hence the map $\text{Ind}^E : \text{Rep}(B) \rightarrow \text{Rep}(A)$ is continuous. In particular, if ${}_A E_B$ is an imprimitivity bimodule with inverse module ${}_B E_A^*$, then Ind^{E^*} gives a continuous inverse to Ind^E , and therefore induction via E induces a homeomorphism between $\text{Rep}(B)$ and $\text{Rep}(A)$ (see [42, Chapter 3.3]).

Basically all induction maps we use in this paper are coming in one way or the other from induction via bimodules, so the above principles can be used. We need the following observation:

Lemma 4.8. *Suppose that (π_i) is a net in $\text{Rep}(K) = \text{Rep}(C^*(K))$ for some compact group K . Then (π_i) converges to some $\sigma \in \widehat{K}$ if and only if there exists an index i_0 such that σ is a sub-representation of π_i for all $i \geq i_0$.*

Proof. Using the Peter-Weyl theorem, we can write $C^*(K) = \bigoplus_{\tau \in \widehat{K}} \text{End}(V_\tau)$. In this picture, given any representation π of K , an irreducible representation τ is a sub-representation of π if and only if the summand $\text{End}(V_\tau)$ is not in the kernel of π , viewed as a representation of $C^*(K)$. Thus weak containment and containment of τ are equivalent.

Assume now that there exists no $i_0 \in I$ such that $\tau \leq \pi_i$ for all $i \geq i_0$. We then construct a subnet (π_j) of (π_i) such that τ is not contained in any of the π_j , which then implies that $\text{End}(V_\tau)$ lies in the kernel of all π_j . But then τ is not weakly contained in the subnet (π_j) which by Lemma 4.7 contradicts $\pi_i \rightarrow \tau$. For the construction of the subnet, we define

$$J := \{(i, k) \in I \times I : k \geq i \text{ and } \tau \not\leq \pi_k\},$$

equipped with the pairwise ordering. The projection to the second factor is clearly order preserving, and if we define $\pi_{(i,k)} := \pi_k$, we obtained a subnet $(\pi_{(i,k)})_{(i,k) \in J}$ with the desired properties. \square

The following proposition will provide the main step towards the proof of Theorem 4.6.

Proposition 4.9. *Suppose that K is a compact group acting on a locally compact space Y and assume that $y \in Y$ is fixed by K . Let $\sigma \in \widehat{K}$ be identified with the representation $\pi_y^\sigma \in (C_0(Y) \rtimes K)^\wedge$ in the usual way and let (y_i, σ_i) be any net in $\text{Stab}(Y)^\wedge$. Then the following are equivalent:*

- (i) *The net $\pi_{y_i}^{\sigma_i} = \text{Ind}_{K_{y_i}}^K(y_i, \sigma_i)$ converges to π_y^σ in $(C_0(Y) \rtimes K)^\wedge$.*
- (ii) *The net (y_i, σ_i) converges to (y, σ) in $\text{Stab}(Y)^\wedge$.*

Recall that for $\sigma \in \widehat{K}$ we denote by $p_\sigma = \dim(V_\sigma)\chi_{\sigma^*}$ the central projection corresponding to σ . Before we give the proof, we should point out a well-known fact on isotypes of unitary representations of compact groups: if $\pi : K \rightarrow U(V_\pi)$ is any such representation, then $\pi(p_\sigma) \in \mathcal{B}(V_\pi)$ is the projection onto the *isotype* V_π^σ of σ in π , that is

$$V_\pi^\sigma = \cup\{V \subseteq V_\pi : \pi|_V \cong \sigma\}.$$

This follows easily from the fact that for any $\tau \in \widehat{K}$ we get $\tau(p_\sigma) = 1_{V_\sigma}$ if $\tau \cong \sigma$ and $\tau(p_\sigma) = 0$ else, and the well known fact that every representation of a compact group decomposes into irreducible ones. In particular, the isotype of an irreducible representation $\sigma \in \widehat{K}$ in the left regular representation $\lambda : K \rightarrow U(L^2(K))$ is the finite dimensional space $V_\sigma \otimes V_\sigma^*$, viewed as a subspace of $L^2(K)$ via the unitary embedding $v \otimes w^* \mapsto \xi_{v,w}$ with $\xi_{v,w}(k) = \sqrt{d_\sigma} \langle v, \sigma(k)w \rangle$ (compare with the proof of Lemma 3.1).

Proof of Proposition 4.9. We may assume without loss of generality that Y is compact, since otherwise we may restrict ourselves to a K -invariant compact neighborhood of y . We first show (i) \Rightarrow (ii). For this consider the canonical inclusion $C^*(K) \rightarrow C(Y) \rtimes K$ which is induced by the K -equivariant embedding $\mathbb{C} \rightarrow C(Y); \lambda \mapsto \lambda 1_Y$. A representation $\pi = P \times U$ of $C(Y) \rtimes K$ restricts to the unitary representation U of K via this inclusion. Thus the map $\text{Rep}(C(Y) \rtimes K) \rightarrow \text{Rep}(K)$ which sends $\pi = P \times U$ to U can be viewed as an induction map via the $C^*(K) - C(Y) \rtimes K$ -bimodule $C(Y) \rtimes K$, and therefore is continuous.

It follows that if $\pi_{y_i}^{\sigma_i} = P_{y_i}^{\sigma_i} \times U_{y_i}^{\sigma_i}$ (we use the notation of §3) converges to $\sigma = \pi_y^\sigma$ as in (i), then $U_{y_i}^{\sigma_i}$ converges to σ in $\text{Rep}(K)$, which by Lemma 4.8 implies that there exists $i_0 \in I$ such that σ is a sub-representation of $U_{y_i}^{\sigma_i}$ for all $i \geq i_0$. But by the Frobenius reciprocity theorem (e.g. see [13, Theorem 7.4.1]) this implies that σ_i is a sub-representation of $\sigma|_{K_{y_i}}$ for all $i \geq i_0$.

It remains to show that (i) implies that $y_i \rightarrow y$. For this we use the canonical inclusion of $C(K \setminus Y)$ into the center of $M(C(Y) \rtimes K)$. The associated “induction map” from $\text{Rep}(C(Y) \rtimes K)$ to $\text{Rep}(C(K \setminus Y))$ sends the representation $\pi_{y_i}^{\sigma_i} = P_{y_i}^{\sigma_i} \times U_{y_i}^{\sigma_i}$ to the restriction of $P_{y_i}^{\sigma_i}$ to $C(K \setminus Y) \subseteq C(Y)$, which is equal to $\text{ev}_{K y_i} \cdot 1_{\mathcal{H}_{\sigma_i}}$. By continuity of “induction” we see that $\text{ev}_{K y_i} \cdot 1_{\mathcal{H}_{\sigma_i}}$ converges to $\text{ev}_{K y} \cdot 1_{V_\sigma}$ in $\text{Rep}(C(K \setminus Y))$ which just means that $K y_i \rightarrow K y = \{y\}$ in $K \setminus Y$. Since K is compact and y is fixed by K , this implies that $y_i \rightarrow y$ in Y .

We now prove (ii) \Rightarrow (i). For this suppose that (y_i, σ_i) is as in (ii). Since every subnet enjoys the same properties, it is enough to show that $\sigma = \pi_y^\sigma$ is weakly contained in $\{\pi_{y_i}^{\sigma_i} : i \in I\}$.

For this let $a \in C(Y) \rtimes K$ be any element such that $\pi_{y_i}^{\sigma_i}(a) = 0$ for all $i \in I$. We need to show that $\pi_y^\sigma(a) = 0$, too. For this recall first from Theorem 2.10 that $C(Y) \rtimes K \cong C(Y \times_K \mathcal{K}(L^2(K)))$ is a continuous C^* -algebra bundle over $K \setminus Y$ with fiber $\mathcal{K}(L^2(K))^{K_{y_i}}$ at the orbit $K y_i$. The projection of $C_0(Y) \rtimes K$ to the fiber $\mathcal{K}(L^2(K))^{K_{y_i}}$ at the orbit $K y_i$ is given via the representation $M_{y_i} \times \lambda$ with λ the regular representation of K and

$$(M_{y_i}(\varphi)\xi)(k) = \varphi(k y_i)\xi(k).$$

It follows from this that the representation of $a \in C(Y) \rtimes K$ on these fibers is given by a net (a_{y_i}) in $\mathcal{K}(L^2(K))^{K_{y_i}} \subseteq \mathcal{K}(L^2(K))$ which converges in the operator norm to the element $a_y \in \mathcal{K}(L^2(K))^K$. At the point y we get the decomposition

$$C^*(K) \cong \mathcal{K}(L^2(K))^K = \bigoplus_{\pi \in \widehat{K}} \mathcal{K}(V_\pi) \otimes 1_{V_\pi^*}$$

and at all other points we get the decompositions

$$\mathcal{K}(L^2(K))^{K_{y_i}} = \bigoplus_{\tau \in \widehat{K}_{y_i}} \mathcal{K}(\mathcal{H}_\tau) \otimes 1_{V_\tau^*},$$

where, for convenience, we write \mathcal{H}_τ for the Hilbert space \mathcal{H}_{U^τ} of U^τ .

In particular, if $p_i : L^2(K) \rightarrow \mathcal{H}_{\sigma_i} \otimes V_{\sigma_i}^* \subseteq L^2(K)$ denotes the orthogonal projection, then the representation $\pi_{y_i}^{\sigma_i} \otimes 1_{V_{\sigma_i}^*} = (P_{y_i}^{\sigma_i} \times U_{y_i}^{\sigma_i}) \otimes 1_{V_{\sigma_i}^*}$ is given by sending the element $a \in C(Y) \rtimes K$ to the element $p_i a_{y_i} p_i \in \mathcal{K}(\mathcal{H}_{\sigma_i}) \otimes 1_{V_{\sigma_i}^*} \subseteq \mathcal{K}(L^2(K))$.

Assume now that $a \in C(Y) \rtimes K$ such that $\pi_{y_i}^{\sigma_i}(a) = 0$ for all $i \in I$. Then all those elements $p_i a_{y_i} p_i$ vanish. We have to show that $\sigma(a_y)$ will vanish, too. For this recall that $\sigma(p_\sigma) = 1_{V_\sigma}$. Thus we may replace a by $p_\sigma a p_\sigma$, where we view p_σ as an element of $C(Y) \rtimes K$ via the embedding $C^*(K) \rightarrow C(Y) \rtimes K$.

Writing $1_i = 1_{V_{\sigma_i}^*}$ we then have

$$\pi_{y_i}^{\sigma_i}(a) \otimes 1_i = \pi_{y_i}^{\sigma_i}(p_\sigma a p_\sigma) \otimes 1_i = (U_{y_i}^{\sigma_i}(p_\sigma) \otimes 1_i)(\pi_{y_i}^{\sigma_i}(a) \otimes 1_i)(U_{y_i}^{\sigma_i}(p_\sigma) \otimes 1_i),$$

where $U_{y_i}^{\sigma_i}(p_\sigma) \otimes 1_i$ is the orthogonal projection from $\mathcal{H}_{\sigma_i} \otimes V_{\sigma_i}^*$ onto the isotype

$$W_i := (\mathcal{H}_{\sigma_i} \otimes V_{\sigma_i}^*)^\sigma = (\mathcal{H}_{\sigma_i} \otimes V_{\sigma_i}^*) \cap (V_\sigma \otimes V_\sigma^*) \subseteq L^2(K).$$

It thus follows that $\pi_{y_i}^{\sigma_i} \otimes 1_i$ represents each $a_i := a_{y_i}$ as an operator on the subspace W_i of the fixed finite dimensional space $V_\sigma \otimes V_\sigma^*$ of $L^2(K)$.

By Frobenius reciprocity, the condition that σ_i is a sub-representation of $\sigma|_{K_{y_i}}$ implies that σ is a sub-representation of the representation $U_{y_i}^{\sigma_i}$, which implies that W_i is non-zero for all $i \in I$. Now let $q_i : V_\sigma \otimes V_\sigma^* \rightarrow W_i$ denote the orthogonal projection. Since $V_\sigma \otimes V_\sigma^*$ is finite dimensional, we may pass to a subnet if necessary to assume that q_i converges to some non-zero projection $q : V_\sigma \otimes V_\sigma^* \rightarrow W$. We then get $0 = \pi_{y_i}^{\sigma_i}(a) \otimes 1_i = q_i a_{y_i} q_i \rightarrow q a_y q \in \mathcal{K}(V_\sigma \otimes V_\sigma^*)$. Moreover, since all W_i are K -invariant, the same is true for W . It follows that the representation $\lambda : C^*(K) \rightarrow \mathcal{K}(L^2(K))^K$ restricts to a multiple of σ on W . But then we have $\sigma(a_y) = 0 \Leftrightarrow q a_y q = 0$ and the result follows. \square

Proof of Theorem 4.6. We first show that the map from $\text{Stab}(X)^\wedge$ to $(C_0(X) \rtimes G)^\wedge$ which assigns (x, σ) to the induced representation π_x^σ is continuous. By the universal property of the quotient topology on $G \backslash \text{Stab}(X)^\wedge$, this will imply that the map

$$\text{Ind} : G \backslash \text{Stab}(X)^\wedge \rightarrow (C_0(X) \rtimes G)^\wedge$$

of Theorem 4.6 is continuous, too. So let (x_i, σ_i) be a net in $\text{Stab}(X)^\wedge$ which converges to some $(x, \sigma) \in \text{Stab}(X)^\wedge$. By the definition of the topology on $\text{Stab}(X)^\wedge$ we may assume that $x_i = g_i y_i$ with $g_i \rightarrow e$ in G , $y_i \in S_x$ (i.e., $G_{y_i} \subseteq G_x$) such that $y_i \rightarrow x$ and $\sigma_i = g_i \tau_i$ with $\tau_i \in \widehat{G}_{y_i}$ such that $\tau_i \leq \sigma|_{G_{y_i}}$. Since induction is constant on G -orbits, we may assume without loss of generality that $y_i = x_i$ and $\tau_i = \sigma_i$. It follows then from Proposition 4.9 that the induced representations $\text{ind}_{G_{x_i}}^{G_x}(x_i, \sigma_i)$ converge to (x, σ) in $(C_0(X) \rtimes G_x)^\wedge$. Using induction in steps and continuity of induction from a fixed subgroup (due to the fact that this induction can be performed via an appropriate Hilbert module), it then follows that

$$\pi_{x_i}^{\sigma_i} = \text{ind}_{G_{x_i}}^G(x_i, \sigma_i) = \text{ind}_{G_x}^G(\text{ind}_{G_{x_i}}^{G_x}(x_i, \sigma_i)) \rightarrow \text{ind}_{G_x}^G(x, \sigma) = \pi_x^\sigma.$$

Conversely, assume that we have a net (π_i) in $(C_0(X) \rtimes G)^\wedge$ which converges to some representation $\pi \in (C_0(X) \rtimes G)^\wedge$. Choose $x_i \in X$ and $\sigma_i \in (C_0(X) \rtimes G)^\wedge$ such that $\pi_i = \pi_{x_i}^{\sigma_i}$, and similarly we realize π as π_x^σ for some $(x, \sigma) \in \text{Stab}(X)^\wedge$. By imbedding $C_0(G \backslash X)$ into $ZM(C_0(X) \rtimes G)$ we can see that $Gx_i \rightarrow Gx$ in $G \backslash X$. Since the quotient map $X \rightarrow G \backslash X$ is open, we can therefore pass to a subnet, if necessary, to assume that $g_i x_i \rightarrow x$ for a suitable net $g_i \in G$. Choosing a local slice Y at x , we may even adjust the situation to guarantee that $g_i x_i \in Y$ for all $i \in I$. Thus, replacing (x_i, σ_i) by $(g_i x_i, g_i \sigma_i)$ we may assume from now on that the net (x_i, σ_i) and the pair (x, σ) all lie in $\text{Stab}(Y)^\wedge$. Using the canonical Morita equivalence $C(G \times_{G_x} Y) \rtimes G \sim_M C_0(Y) \rtimes G_x$ together with Proposition 3.13 it follows that $\tau_{x_i}^{\sigma_i} = \text{Ind}_{G_{x_i}}^{G_x}(x_i, \sigma_i)$ converges to $\tau_x^\sigma = \sigma$ in $(C_0(Y) \rtimes G_x)^\wedge$. It is then a consequence of Proposition 4.9 that (x_i, σ_i) converges to (x, σ) in $\text{Stab}(Y)^\wedge$. But this also implies convergence in $\text{Stab}(X)^\wedge$ and thus of the respective orbits in $G \backslash \text{Stab}(X)^\wedge$. \square

Before discussing an example, we emphasize the following consequences of the above discussion.

- For a proper action of a discrete group G on X , a subset $U \subset \text{Stab}(X)^\wedge$ is open if and only if the following is true: if $(x, \pi) \in U$, then there exists an open slice V_x around x such that U contains all points (y, τ) such that $y \in V_x$ and $\tau \leq \pi|_{G_y}$.
- In any case, the set $X \cong \{(x, 1_{G_x}) : x \in X\} \subset \text{Stab}(X)^\wedge$ is open in this topology, and by projection, $G \backslash X$ (identified with $G \backslash \{(x, 1_{G_x}) : x \in X\}$) is open in $G \backslash \text{Stab}(X)^\wedge$.

Example 4.10. We come back to our example $C(\mathbb{T}^2) \rtimes G$ with $G = D_4 = \langle R, S \rangle$ as discussed in Examples 3.5, 3.5 and 3.12. Consider the topological fundamental domain

$$Z := \{(e^{2\pi i s}, e^{2\pi i t}) : 0 \leq t \leq \frac{1}{2}, 0 \leq s \leq t\} \subseteq \mathbb{T}^2$$

for the action of G on \mathbb{T}^2 . Recall that this means that the canonical map from Z to $G \backslash \mathbb{T}^2$ is a homeomorphism. It then is easily checked that

$$\text{Stab}(Z)^\wedge = \{(z, w), \sigma\} \in \text{Stab}(\mathbb{T}^2)^\wedge : (z, w) \in Z\}$$

is a topological fundamental domain for the action of G in $\text{Stab}(\mathbb{T}^2)^\wedge$.

Thus, in order to describe the topology of $C_0((\mathbb{T}^2) \rtimes G)^\wedge \cong G \backslash \text{Stab}(\mathbb{T}^2)^\wedge$ it suffices to describe the topology of $\text{Stab}(Z)^\wedge$. In what follows we identify Z with the triangle

$$\{(s, t) \in \mathbb{R}^2 : 0 \leq t \leq \frac{1}{2}, 0 \leq s \leq t\}.$$

In Example 3.5 we already computed the stabilizers and their representations.

Each point in the interior $\overset{\circ}{Z}$ has trivial stabilizer $\{E\}$. Since the trivial group has only the trivial representation and since every representation of a group restricts obviously to a multiple of the trivial representation of the trivial group, we see that if $(s_n, t_n)_{n \in \mathbb{N}}$ is a sequence in $\overset{\circ}{Z}$ which converges to some $(s, t) \in Z$, then $((s_n, t_n), 1_{\{E\}})$ converges to any $((s, t), \sigma)$ with $\sigma \in \widehat{G}_{(s, t)}$. If (s_n, t_n) converges to 0, then the sequence $((s_n, t_n), 1_{\{E\}})$ has the five limit points $((0, 0), 1_G), ((0, 0), \chi_1), ((0, 0), \chi_2), ((0, 0), \chi_3), ((0, 0), \lambda)$ with $\{1_G, \chi_1, \chi_2, \chi_3, \lambda\}$ as defined in Example 3.5.

Let us now restrict our attention to the three border lines. For example, if we consider the line segment $I_1 := \{(s, s) : 0 < s < \frac{1}{2}\}$ we have the constant stabilizer $K_1 = \langle RS \rangle$ and we see that

$$\{((s, s), \sigma) \in \text{Stab}(Z)^\wedge : (s, s) \in I_1\} = I_1 \times \{1_{K_1}, \epsilon_{K_1}\}$$

topologically. Similar descriptions hold for the line segments $I_2 := \{(0, t) : 0 < t < \frac{1}{2}\}$ and $I_3 := \{(s, \frac{1}{2}) : 0 < s < \frac{1}{2}\}$.

In order to describe the topology of $\text{Stab}(Z)^\wedge$ at the corners $(0, 0)$, $(\frac{1}{2}, \frac{1}{2})$ and $(0, \frac{1}{2})$ of Z , we observe that

$$\begin{aligned} 1_{K_1} &\leq \text{res}_{K_1}^G(1_G), \text{res}_{K_1}^G(\chi_2), \text{res}_{K_1}^G(\lambda), \text{ and } \epsilon_{K_1} \leq \text{res}_{K_1}^G(\chi_1), \text{res}_{K_1}^G(\chi_3), \text{res}_{K_1}^G(\lambda), \\ 1_{K_2} &\leq \text{res}_{K_2}^G(1_G), \text{res}_{K_2}^G(\chi_3), \text{res}_{K_2}^G(\lambda), \text{ and } \epsilon_{K_2} \leq \text{res}_{K_2}^G(\chi_1), \text{res}_{K_2}^G(\chi_2), \text{res}_{K_2}^G(\lambda), \\ 1_{K_3} &\leq \text{res}_{K_3}^G(1_G), \text{res}_{K_3}^G(\chi_3), \text{res}_{K_3}^G(\lambda), \text{ and } \epsilon_{K_3} \leq \text{res}_{K_3}^G(\chi_1), \text{res}_{K_3}^G(\chi_2), \text{res}_{K_3}^G(\lambda), \end{aligned}$$

which implies, for example, that if a sequence $(s_n, s_n)_{n \in \mathbb{N}} \subseteq I_1$ converges to $(0, 0)$ the sequence $((s_n, s_n), 1_{K_1})$ has the limit points $((0, 0), 1_G)$, $((0, 0), \chi_2)$, $((0, 0), \lambda)$ and the sequence $((s_n, s_n), \epsilon_{K_1})$ has the limit points $((0, 0), \chi_1)$, $((0, 0), \chi_3)$, $((0, 0), \lambda)$. Similar descriptions follow from the above list for sequences in I_1, I_2, I_3 which converge to any of the corners $(0, 0)$ or $(\frac{1}{2}, \frac{1}{2})$.

At the corner $(0, \frac{1}{2})$ with stabilizer $H = \langle R^2, S \rangle$ and characters $1_H, \mu_1, \mu_2, \mu_3$ of H , as described in Example 3.5, we get the relations

$$\begin{aligned} 1_{K_2} &= \text{res}_{K_2}^H(1_H), \text{res}_{K_2}^H(\mu_2), \text{ and } \epsilon_{K_2} = \text{res}_{K_2}^H(\mu_1), \text{res}_{K_2}^H(\mu_3), \\ 1_{K_3} &= \text{res}_{K_3}^H(1_H), \text{res}_{K_3}^H(\mu_3), \text{ and } \epsilon_{K_3} = \text{res}_{K_3}^H(\mu_1), \text{res}_{K_3}^H(\mu_2), \end{aligned}$$

which implies, for example, that for a sequence $(s_n, \frac{1}{2})$ in I_3 which converges to $(0, \frac{1}{2})$, the sequence $((s_n, \frac{1}{2}), 1_{K_3})$ has the limit points $((0, \frac{1}{2}), 1_H)$, $((0, \frac{1}{2}), \mu_3)$ and the sequence $((s_n, \frac{1}{2}), \epsilon_{K_3})$ has the limit points $((0, \frac{1}{2}), \mu_1)$, $((0, \frac{1}{2}), \mu_2)$.

From this description we obtain an increasing sequence of open subsets of $\text{Stab}(Z)^\wedge$

$$\emptyset = \mathcal{O}_0 \subseteq \mathcal{O}_1 \subseteq \mathcal{O}_2 \subseteq \mathcal{O}_3 = \text{Stab}(Z)^\wedge$$

(and corresponding open subsets of $(C_0(\mathbb{T}^2) \rtimes G)^\wedge$ via induction) with

$$\begin{aligned} \mathcal{O}_1 &= \{((x, y), 1) \mid (x, y) \in Z\} \cong Z \\ \mathcal{O}_2 \setminus \mathcal{O}_1 &= \{((s, s), \epsilon_{K_1}) : 0 < s < 1/2\} \cup \{((0, t), \epsilon_{K_2}) : 0 < t < 1/2\} \\ &\quad \cup \{((s, 1/2), \epsilon_{K_3}) : 0 < s < 1/2\} \\ &\quad \cup \{((0, 0), \chi_1), ((1/2, 1/2), \chi_1), ((0, 1/2), \mu_1)\} \cong \partial Z \\ \mathcal{O}_3 \setminus \mathcal{O}_2 &= \{(0, 0), (1/2, 1/2)\} \times \{\chi_2, \chi_3, \lambda\} \cup \{(0, 1/2)\} \times \{\mu_2, \mu_3\} \cong \{1, \dots, 8\}. \end{aligned}$$

Thus we obtain a corresponding sequence of ideals

$$\{0\} = I_0 \subseteq I_1 \subseteq I_2 \subseteq I_3 = C_0(\mathbb{T}^2) \rtimes G$$

with

$$\widehat{I_1} \cong Z, \quad \widehat{I_2/I_1} \cong \partial Z, \quad \widehat{I_3/I_2} \cong \{1, \dots, 8\}.$$

Indeed, if we combine the above description of the topology of $\text{Stab}(Z)^\wedge \cong (C(\mathbb{T}^2) \rtimes G)^\wedge$ with the description of the algebra

$$C(\mathbb{T}^2) \rtimes G \cong \{f \in C(Z, \mathcal{K}(\ell^2(G))) : f(s, t) \in \mathcal{K}(\ell^2(G))^{G_{(s,t)}}\}$$

as given in Example 3.5, we see that $I_1 = I_{\mathbb{T}^2}$ as described in Example 3.12. It is Morita-equivalent to $C(Z) \cong C(G \backslash \mathbb{T}^2)$ and consists of those functions $f \in C(Z, \mathcal{K}(\ell^2(G)))$ which only take non-zero values in the matrix blocks corresponding to the trivial representations in each fiber $\mathcal{K}(\ell^2(G))^{G_{(s,t)}}$. The ideal I_2 takes arbitrary values in $\mathcal{K}(\ell^2(G))^{G_{(s,t)}}$ at each $(s, t) \in Z \setminus \{(0, 0), (\frac{1}{2}, \frac{1}{2}), (0, \frac{1}{2})\}$ and it takes non-zero values only in the diagonal entries d_1, d_{χ_1} at the corners $(0, 0)$ and $(\frac{1}{2}, \frac{1}{2})$, with respect to the block decomposition of $\mathcal{K}(\ell^2(G))^G$ as given in Example 3.5, and it takes only non-zero values in the 2×2 block entries B_1, B_{μ_1} at the corner $(0, \frac{1}{2})$. It is then clear that I_2/I_1 is isomorphic to the algebra of continuous functions $f : \partial Z \rightarrow M^8(\mathbb{C})$ such that on the line segments I_i , $i = 1, 2, 3$, $f(s, t)$

has non-zero entries only in the 4×4 matrix blocks $A_{\epsilon_{K_i}}$, at the corners $(0,0)$ and $(\frac{1}{2}, \frac{1}{2})$ it has non-zero entry only in the diagonal entry d_{χ_1} and at the corner $(0, \frac{1}{2})$ it has non-zero entries only in the 2×2 block C_{μ_1} . It follows that I_2/I_1 is Morita equivalent to $C(\partial Z)$. Finally, the quotient $(C(\mathbb{T}^2) \rtimes G)/I_2$ is isomorphic to $M_2(\mathbb{C})^4 \oplus \mathbb{C}^4$.

Remark 4.11. If G acts freely and properly on X , then Green's theorem ([29]) implies that $C_0(X) \rtimes G \cong C_0(X \times_G \mathcal{K})$ is Morita equivalent to $C_0(G \backslash X)$. If everything in sight is second countable, this implies that the bundle $C_0(X \times_G \mathcal{K})$ is (stably) isomorphic to the trivial bundle $C_0(G \backslash X, \mathcal{K})$ (after stabilization, if necessary, we may assume $\mathcal{K} \cong \mathcal{K}(\ell^2(\mathbb{N}))$ everywhere).

So one may wonder, whether a similar result can be true in general, i.e., is there a chance to show that for any proper action of G on a locally compact space X the bundle $C_0(X \times_G \mathcal{K})$ is stably isomorphic to a subbundle of the trivial bundle $C_0(G \backslash X, \mathcal{K})$, so that the fibre over the orbit Gx would be a suitable subalgebra of $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}))$ stably isomorphic to $\mathcal{K}(L^2(G))^{\text{Ad } \rho(G_x)}$. Actually it follows from Proposition 2.7 that this is always the case if there exists a topological fundamental domain $Z \cong G \backslash X$ for the action of G on X as in Definition 2.4.

Unfortunately, such a trivialization is not possible in general. Indeed, the problem already appears in the case of linear actions of finite groups on the closed unit ball in \mathbb{R}^n . We shall present a concrete counter-example in the following section.

5. K-THEORY OF PROPER ACTIONS

In this chapter we will consider the problem of calculating equivariant K-theory for proper actions.

Definition 5.1. Let X be a proper G -space where G is a locally compact group. The G -equivariant K-theory of X , denoted $K_G^*(X)$, is the K-theory of the crossed product C^* -algebra $C_0(X) \rtimes G$.

There are very few *general* results about equivariant K-theory. The ones we discuss below all treat simplifications of the problem. There are several kinds of simplifications possible. One involves ignoring torsion in $K_G^*(X)$. Thus, one can aim for a computation of equivariant K-theory with rational coefficients $K_G^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. For technical purposes it is often convenient to tensor with \mathbb{C} instead, this gives equivalent results. Another simplification is to consider only compact groups. A further simplification is to restrict to finite groups.

We start with the question of whether or not equivariant K-theory can be described, as with non-equivariant K-theory, in terms of vector bundles.

Let G be possibly non-compact, locally compact group, acting properly on X with compact space $G \backslash X$ of orbits. Let E be a G -equivariant vector bundle on X . Because the action is proper, there is a G -invariant Hermitian structure on E . We can equip the space $\Gamma_c(E)$ of continuous sections of E with compact supports with the right $C_c(G \times X)$ -module structure via

$$(s \cdot a)(x) = \int_G g \cdot s(g^{-1}x) a(g^{-1}, g^{-1}x) \sqrt{\Delta(g^{-1})} dg, \quad s \in \Gamma_c(E), a \in C_c(G \times X)$$

where we regard $C_c(G \times X)$ as a dense subalgebra of $C_0(X) \rtimes G$. One can also define a right $C_0(X) \rtimes G$ -valued inner product on $\Gamma_0(E)$, by the formula

$$(5.1) \quad \langle s_1, s_2 \rangle(x, g) := \sqrt{\Delta(g^{-1})} \langle s_1(x), g \cdot s_2(g^{-1}x) \rangle_E.$$

So the completion $\tilde{\Gamma}(E) := \overline{\Gamma_c(E)}$ with respect to this inner product becomes a right $C_0(X) \rtimes G$ -Hilbert module. If $G \backslash X$ is compact, this Hilbert module can be checked to be finitely generated. The co-compactness assumption implies that the

identity operator is a compact Hilbert module operator. See *e.g.* [24] for a proof, and the following example for the case where $E = 1_X$ is the trivial line bundle.

Example 5.2. Let G be locally compact, and act properly and co-compactly on X . If E is the trivial line bundle 1_X over X , with the trivial action of G on the fibres then the finitely generated projective Hilbert $C_0(X) \rtimes G$ -module $\tilde{\Gamma}(1_X)$ as above is isomorphic to the range of any of the idempotents $p_X \in C_0(X) \rtimes G$, constructed by a cut-off function c . Recall that, since $G \backslash X$ is compact, $c : X \rightarrow [0, 1]$ is a compactly supported continuous function such that $\int_G c(g^{-1}x)^2 dg = 1$ for all $x \in X$. Then $p_X := \langle c, c \rangle$ is a projection in $C_0(X) \rtimes G$ (compare with the discussion around (3.6)) and we get

$$\tilde{\Gamma}(1_X) \cong p_X \cdot (C_0(X) \rtimes G)$$

as right $C_0(X) \rtimes G$ -modules. For the proof, regard c as an element of $\tilde{\Gamma}(1_X)$. Define

$$Q : \tilde{\Gamma}(1_X) \rightarrow C_0(X) \rtimes G, \quad Q(s) := \langle c, s \rangle, \quad R : C_0(X) \rtimes G \rightarrow \tilde{\Gamma}(1_X), \quad R(a) := ca,$$

using the inner product in (5.1). Then for all $s \in \tilde{\Gamma}(1_X)$, $a \in C_0(X) \rtimes G$,

$$RQ(s) = c \cdot \langle c, s \rangle = s, \quad \text{and} \quad QR(a) = \langle c, ca \rangle = p_X a.$$

In particular this shows that $\tilde{\Gamma}(1_X)$ is a rank-one module. It also gives another proof that the K-theory classes $[p_X] \in K_G^0(X)$ are independent of the choice of cut-off function. Note also that the natural representation of $C(G \backslash X) \rightarrow \mathcal{L}_{C_0(X) \rtimes G}(\tilde{\Gamma}(1_X))$ by letting G -invariant functions act as multiplication operators on $\tilde{\Gamma}(1_X)$, gives an isomorphism

$$C(G \backslash X) \cong \mathcal{K}(\tilde{\Gamma}(1_X)).$$

Consequently, we get an isomorphism

$$C(G \backslash X) \cong \mathcal{K}(p_X \cdot C_0(X) \rtimes G) \cong p_X \cdot C_0(X) \rtimes G \cdot p_X$$

(see the discussion around (3.7).)

Let $\text{VK}_G^0(X)$ be the Grothendieck group of the monoid of G -equivariant complex vector bundles on X . Let $\text{VK}_G^1(X)$ be defined to be the kernel of the map $\text{VK}_G^0(X \times S^1) \rightarrow \text{VK}_G^0(X)$ by restriction of G -equivariant vector bundles to $X \times \{1\}$, where we let G act trivially on S^1 . Then application of the above procedure to cycles yields a map

$$(5.2) \quad \text{VK}_G^*(X) \rightarrow K_*(C_0(X) \rtimes G).$$

Is (5.2) an isomorphism in general? Corollary 5.3 of [24] shows that for any proper action of a locally compact group G , the monoid of isomorphism classes of G -equivariant vector bundles on X is isomorphic to the monoid of projections in $(C_0(X) \rtimes G) \otimes \mathcal{K}$. However, the K-theory of $C_0(X) \rtimes G$ is defined in terms of the monoid of projections in $(C_0(X) \rtimes G)^+ \otimes \mathcal{K}$, where the $+$ denotes unitization, so that *a priori* there could be classes which do not come from equivariant vector bundles. In fact, the question is somewhat delicate. See the analysis [24], following work of Lück and Oliver in [37], who are partially responsible for the following result.

Theorem 5.3. *If G is discrete, compact or an almost-connected group, and if X is G -compact, then the map (5.2) is an isomorphism.*

For groups not satisfying one of the conditions of Theorem 5.3, equivariant vector bundles need not generate all of equivariant K-theory.

A counter-example to isomorphism of (5.2) can be built using the following ideas, essentially due to Julianne Sauer. Let G be the semi-direct product $\mathbb{T}^2 \rtimes_A \mathbb{Z}$

where \mathbb{Z} acts by the automorphism of the compact group \mathbb{T}^2 induced by a matrix $A \in \mathrm{GL}(2, \mathbb{Z})$. Then G acts properly on \mathbb{R} with \mathbb{Z} acting by translations and \mathbb{T}^2 acting trivially. The restriction of any G -equivariant vector bundle on \mathbb{R} to a point of \mathbb{R} determines a finite-dimensional representation of the compact subgroup $\mathbb{T}^2 \subset G$. It can be shown that if A is a hyperbolic matrix, then only multiples of the trivial representation can be obtained in this way. This shows that there are rather few G -equivariant vector bundles on \mathbb{R} . Of course G is neither compact, nor a Lie group.

Equivariant Kasparov theory offers another point of view on equivariant K-theory for G -compact spaces. The *representable* G -equivariant K-theory of a proper G -space X – denoted $\mathrm{RK}_G^*(X)$ – is defined in terms of G -equivariant maps from X to suitable spaces of Fredholm operators on G -Hilbert spaces; such maps may always be normalized so that the Hilbert space may be taken fixed, so the representable K-theory has cycles maps $X \rightarrow \mathrm{Fred}(H_G)$. An equivariant version of the Kuiper theorem implies such maps may be taken norm continuous. For G -compact spaces, it can be shown that $\mathrm{RK}_G^*(X) \cong \mathrm{K}_G^*(X)$ (see [24, Theorem 3.8]). This gives a reasonably satisfactory homotopy-theoretic description of equivariant K-theory for some purposes, but it is not very concrete.

For general proper actions of locally compact groups G on spaces X , $\mathrm{K}_G^*(X)$ is a G -compactly supported theory and $\mathrm{RK}_G^*(X)$ is not, so the theories definitely differ in this case.

Remark 5.4. Unlike non-equivariant K-theory, which is rationally isomorphic to cohomology, equivariant K-theory is definitely different from equivariant *cohomology* – even for finite groups, and even after tensoring both with the complex numbers. Equivariant cohomology for a compact group action is defined to be the ordinary cohomology of $X \times_G EG$, where EG is the classifying space for *free* actions of G . It is not hard to check that equivariant cohomology with *complex* (or rational) coefficients for *finite* groups G is the same as the G -invariant part of ordinary cohomology, and hence (by Proposition 5.5 below) it agrees with just the ordinary cohomology of the quotient space, whereas equivariant K-theory does not behave like this, even for finite groups, for already integrally $\mathrm{K}_G^*(\mathrm{pt}) \cong \mathrm{Rep}(G)$ is free abelian with one generator for each irreducible representation of G . What is true is that equivariant cohomology with complex coefficients and *finite* groups, is isomorphic to the *localization* of equivariant K-theory with complex coefficients at the identity conjugacy class of the group, *c.f.* [7]). Equivariant K-theory with complex coefficients may thus be viewed as ‘de-localized’ equivariant cohomology – the point of view taken in the article [7] of Baum and Connes, and explained below in §5.1.

We start by justifying one of the statements made in the above Remark.

Proposition 5.5. *For any G -space X , G finite,*

$$(5.3) \quad \mathrm{K}^*(X)^G \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{|G|}] \cong \mathrm{K}^*(G \backslash X) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{|G|}]$$

as $\mathbb{Z}[\frac{1}{|G|}]$ -modules. The isomorphism is induced by the pull-back map $\pi^: \mathrm{K}^*(G \backslash X) \rightarrow \mathrm{K}^*(X)$. In particular, $X \mapsto \mathrm{K}^*(X)^G$ and $X \mapsto \mathrm{K}^*(G \backslash X)$ agree for finite groups, after tensoring by \mathbb{C} .*

Proposition 5.5 does not hold for compact groups in general. It is easy to find a counter-example: take the \mathbb{T} -action on the 2-sphere which rotates around the z -axis. The quotient space is the closed interval $[0, 1]$, so $\mathrm{K}^*(\mathbb{T} \backslash S^2) \cong \mathbb{Z}$. But since \mathbb{T} is connected, it acts trivially on K-theory of S^2 , so $\mathrm{K}^*(S^2)^{\mathbb{T}} \cong \mathrm{K}^*(S^2) \cong \mathbb{Z} \oplus \mathbb{Z}$. These are obviously different even after tensoring with \mathbb{C} .

Nor does the Proposition hold for infinite discrete groups, for a different reason. If G is infinite and $X = G$ with G acting by translation, then $K^*(G \backslash X) \cong K_*(\mathbb{C}) \cong \mathbb{Z}$ but since there are no nonzero finitely supported G -invariant maps $G \rightarrow \mathbb{Z}$, and since K -theory is compactly supported, $K^*(X)^G = 0$.

Proof of Proposition 5.5. Closed G -invariant subspaces of X are in 1-1 correspondence with closed subspaces of $G \backslash X$. If $Z \subset X$ is a closed invariant subspace, then the exact sequence

$$0 \rightarrow C_0(X \setminus Z) \rightarrow C_0(X) \rightarrow C_0(Z) \rightarrow 0$$

of G - C^* -algebras induces a long exact sequence of G -equivariant K -theory groups

$$(5.4) \quad \cdots \rightarrow K^*(G \backslash Z) \rightarrow K^*(G \backslash X) \rightarrow K^*(G \backslash Z) \xrightarrow{\partial} K^{*+1}(G \backslash (X \setminus Z)) \rightarrow \cdots$$

and also an exact sequence

$$\cdots \rightarrow K^*(X \setminus Z) \rightarrow K^*(X) \rightarrow K^*(Z) \xrightarrow{\partial} K^{*+1}(X \setminus Z) \rightarrow \cdots$$

of non-equivariant groups, each of which carries an action of G , which therefore induces a long exact sequence

$$(5.5) \quad \cdots K^*(X)^G \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{|G|}] \rightarrow K^*(Z)^G \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{|G|}] \xrightarrow{\partial} K^{*+1}(X \setminus Z)^G \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{|G|}] \rightarrow \cdots$$

of the G -invariant elements. Note that we have to tensor with $\mathbb{Z}[\frac{1}{|G|}]$ to guarantee that this sequence is exact. The map π^* of the theorem induces a map between the sequence (5.4), tensored by $\mathbb{Z}[\frac{1}{|G|}]$, and sequence (5.5), so it follows from the Five-lemma that π^* is an isomorphism for X , if this is true for Z and $X \setminus Z$.

We now proceed by induction on $|G|$. If $Z \subseteq X$ is the closed set of G -fixed points, we have $K^*(G \backslash X) = K^*(X) = K^*(X)^G$, so by the above discussion we may assume that all stabilizers for the action of G on X are proper subgroups of G . We then find a cover \mathcal{U} of X by invariant open sets U each isomorphic to some induced G -space $G \times_H Y$ with $|H| < |G|$.

For each such set $U \cong G \times_H Y$ we have $K^*(G \backslash U) \cong K^*(H \backslash Y)$. On the other hand $G \times_H Y$ fibres by the first coordinate projection over the finite G -set G/H ; the fibre over gH is Y . Hence the ordinary K -theory of U decomposes as a direct sum

$$K^*(G \times_H Y) \cong \bigoplus_{gH \in G/H} K^*(Y).$$

The group G permutes the summands by left translation and the invariant part $K^*(G \times_H Y)^G$ is isomorphic to $K^*(Y)^H$. By induction, the formula of the lemma is true for H , and since $|H|$ divides $|G|$ we get

$$K^*(U)^G \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{|G|}] \cong K^*(G \backslash U) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{|G|}].$$

Now suppose a G -space X is covered by n induced open sets U_1, \dots, U_n . Put $Z = X \setminus U_n$. Then Z is covered by the $n-1$ open sets U_1, \dots, U_{n-1} . By Meyer-Vietoris and the above discussion for induced sets, the lemma holds for X if it holds for Z . Thus by induction on n we may therefore assume the lemma to be proved for all G -sets which are covered by finitely many properly induced open sets. The result then follows from the fact that the theories $K^*(X)^G$ and $K^*(G \backslash X)$ both respect inductive limits. \square

We are going to start our investigation of equivariant K -theory with a summary of what is known about finite group actions. We will also neglect torsion for the moment – in fact, it will be helpful to tensor all equivariant K -theory groups in the following by \mathbb{C} , since this has the consequence that if G is finite, then $\text{Rep}(G) \otimes_{\mathbb{Z}} \mathbb{C}$,

as a ring, is simply the ring of complex valued functions on the set of conjugacy classes in G (this would not be true even if we tensored by \mathbb{Q} .) Now $K_G^*(X)$ is always a module over $\text{Rep}(G)$. Tensoring everything by \mathbb{C} then gives $K_G^*(X) \otimes_{\mathbb{Z}} \mathbb{C}$ the structure of a module over $\text{Rep}(G) \otimes_{\mathbb{Z}} \mathbb{C}$, and any module over the ring of complex-valued functions on a finite set of points decomposes as a direct sum over the spectrum of the ring (which in this case is the set of conjugacy classes in G .) This provides some additional algebraic structure which proves to be very useful.

In what follows, we write $K_G^*(X)_{\mathbb{C}} := K_G^*(X) \otimes_{\mathbb{Z}} \mathbb{C}$ for the usual integral G -equivariant K-theory of X , tensored by the complex numbers and we write $\text{Rep}(G)_{\mathbb{C}}$ for $\text{Rep}(G) \otimes_{\mathbb{Z}} \mathbb{C}$.

We start by simplifying even further, and discuss the *difference* of the ranks of $K_G^0(X)_{\mathbb{C}}$ and $K_G^1(X)_{\mathbb{C}}$. This integer is called the *equivariant Euler characteristic* of X . Denote the equivariant Euler characteristic by $\text{Eul}(G \ltimes X)$. Although a crude invariant, it is at least easily geometrically computable, by a version of the Lefschetz fixed-point theorem (proved by Atiyah, Theorem 5.9 below.)

Clearly the equivariant Euler characteristic of a free action is the ordinary Euler characteristic of the quotient space. On the other hand, the Euler characteristic is multiplicative under coverings, so for a free action of a finite group

$$\text{Eul}(G \ltimes X) = \text{Eul}(G \backslash X) = \frac{1}{|G|} \text{Eul}(X).$$

The following lemma describes $\chi(G \backslash X)$ geometrically, even in the presence of isotropy, with the additional hypothesis of a smooth action on a smooth manifold.

Lemma 5.6. *Let X be a smooth compact manifold and G a finite group acting smoothly on X . Then*

$$(5.6) \quad \text{Eul}(G \backslash X) = \frac{1}{|G|} \sum_{g \in G} \text{Eul}(X^g),$$

where X^g is the fixed-point submanifold of g .

Proof. Using the standard formula for the dimension of the space of G -invariants in a representation and the fact that $K^*(G \backslash X) \otimes_{\mathbb{Z}} \mathbb{C}$ is the same as the G -invariant part of $K^*(X) \otimes_{\mathbb{Z}} \mathbb{C}$,

$$\text{Eul}(G \backslash X) = \frac{1}{|G|} \sum_{g \in G} \chi^{\text{Eul}}(g)$$

where χ^{Eul} is the (virtual) character of the $\mathbb{Z}/2$ -graded representation of G on $K^*(X)_{\mathbb{C}} (\cong H^*(X, \mathbb{C}))$, that is, the difference of the characters of G acting on even and odd K-theory tensored with \mathbb{C} , or cohomology with coefficients in \mathbb{C} .

Now by the Lefschetz fixed-point formula, $\chi^{\text{Eul}}(g) := \text{trace}_s(g) = \text{Eul}(X^g)$, which proves the result. \square

In the last proof, we used the very strong version of the Lefschetz fixed-point formula applicable to an *isometry* of a compact Riemannian manifold. This classical fact seems quite well-known to topologists, but for lack of a reference, we cite the second author's article [22] for this (see Theorem 10. For the connection between the Lefschetz map and traces, see [21].) It also follows immediately from the Atiyah-Segal-Singer index theorem (see Theorem 2.12 of [4].)

The following gives a number of examples of interesting finite group actions on surfaces.

Example 5.7. This example is due P.E. Conner and E.E. Floyd in [12]. Let p and q be two odd primes, let $n = pq$ and λ be a primitive n th root of unity. Let $S \subset \mathbb{C}P^2$ be the zero locus of the homogeneous polynomial $f(z_1, z_2, z_3) = z_1^n + z_2^n + z_3^n$. Then

S is a smooth complex submanifold of \mathbb{CP}^2 of complex dimension 1, *i.e.* is a curve. Let $T([z_1, z_2, z_3]) := [z_1, \lambda^p z_2, \lambda^q z_3]$. Then T has order n . Let $\Gamma = \langle T \rangle$. Let

$$\tau: S \rightarrow S, \tau([z_1, z_2, z_3]) := [\lambda z_1, z_2, z_3].$$

It commutes with Γ , has order n . Thus we get an action of $\mathbb{Z}/n \times \mathbb{Z}/n$ on S . This is the restriction of an action on \mathbb{CP}^2 of course. The fixed-point set of τ in \mathbb{CP}^2 is points with first homogeneous coordinate zero, so it is a copy of \mathbb{CP}^1 . The quotient $X := \Gamma \backslash S$ is a closed Riemann surface. Its genus is deduced from its Euler characteristic, which by the formula above is given by $\frac{1}{n} \sum_{0 \leq j < n} \text{Eul}(X^{T^j})$. Each T^j has a finite number of fixed-points, and, counting them, one computes that $\chi(X) = 2 - (p-1)(q-1)$ and hence the genus of $X := G \backslash S$ is $\frac{(p-1)(q-1)}{2}$.

The map

$$\tau: S \rightarrow S, \tau([z_1, z_2, z_3]) := [\lambda z_1, z_2, z_3]$$

descends to a self-map $\dot{\tau}: X \rightarrow X$. We obtain an action of the group $G := \mathbb{Z}/n$ on a surface, such that exactly one point has non-trivial isotropy.

Remark 5.8. There are simpler examples of a finite group action on a compact space with exactly one fixed point. Take a rotation of the Euclidean plane V , extend this uniquely to an orientation-preserving isometry of $V \oplus \mathbb{R}$, and consider the induced map on the projectivisation $\mathbb{P}(V \oplus \mathbb{R}) \cong \mathbb{RP}^2$. This fixes exactly one point (the point at infinity.)

By contrast, the *equivariant* Euler characteristic is more complicated. Atiyah and Segal proved the following result about it in [5]. The proof is more or less immediate from the Lefschetz fixed-point theorem and a theorem of Baum and Connes in [7] discussed in §5.1.

Theorem 5.9 (see [5]). *Let G be a finite group acting smoothly on a smooth compact manifold X . Then*

$$\text{Eul}(G \ltimes X) = \frac{1}{|G|} \sum_{g_1, g_2} \text{Eul}(X^{g_1, g_2}),$$

where the sum is over all commuting pairs of group elements, and where X^{g_1, g_2} denotes the joint fixed-point set of g_1, g_2 .

This local expression for the equivariant, or *orbifold* Euler characteristic, had already appeared in the physics literature in connection with string theory before Atiyah and Segal interpreted it in terms of equivariant K-theory in [5].

5.1. The theorem of Baum and Connes. *In this subsection, $K_G^*(X)_{\mathbb{C}}$ continues to denote K-theory with complex coefficients. If G is a group, $[G]$ will denote the set of conjugacy classes in G .*

Let G be finite, X a G -space. Then, as mentioned above, the equivariant K-theory $K_G^*(X)$ is a module over the representation ring $\text{Rep}(G)$, and similarly for the complexifications. This fact is the main ingredient to the arguments which follow.

The complex representation ring $\text{Rep}(G)_{\mathbb{C}}$ is naturally isomorphic to the ring of complex class functions on G , *i.e.* the ring of functions on the finite set $[G]$ of conjugacy classes in G . The isomorphism is given by sending an irreducible representation $\tau \in \hat{G}$ to its character $\chi_{\tau} = \text{trace } \tau$.

Any module M over a ring of this form, *i.e.* of the form $\mathbb{C}^{[G]}$ with pointwise multiplication and addition, decomposes into a direct sum of modules $M_{[g]}$, one component for each summand, *i.e.* conjugacy class, $[g] \in [G]$. The components are the *localizations* of M at these conjugacy classes:

$$M_{[g]} = M \otimes_{\text{Rep}(G)} \mathbb{C}$$

where \mathbb{C} is understood as a left $\text{Rep}(G)$ -module by evaluation of characters at $[g]$.

Remark 5.10. If G is a finite group and X is a trivial G -space, then it follows from [47, Proposition 2.2] that $K_G^*(X) \cong K^*(X) \otimes \text{Rep}(G)$, where the isomorphism sends the class $[V]$ of a G -equivariant vector bundle V in $K_G^0(X)$ (assuming w.l.o.g. that X is compact) to the sum $\sum_{\tau \in \widehat{G}} [\text{Hom}^G(V_\tau, V) \otimes V_\tau] \in K_0(X) \otimes \text{Rep}(G)$, where V_τ denotes the representation space of τ and $\text{Rep}(G)$ is regarded as the free abelian group with generators the irreducible representations of G (recall that $\dim(\text{Hom}(V_\tau, V)_x)$ gives the multiplicity of τ in the fiber V_x of V at $x \in X$). After complexification and using the above explained realization of $\text{Rep}(G)_\mathbb{C}$ as $\mathbb{C}^{[G]}$ we obtain an isomorphism

$$\Phi^0 : K_G^0(X)_\mathbb{C} \xrightarrow{\cong} K^0(X)_\mathbb{C} \otimes \mathbb{C}^{[G]} \cong \bigoplus_{[g] \in [G]} K^0(X)_\mathbb{C}$$

given on the class of an equivariant vector bundle $[V] \in K_G^0(X)$ by

$$[V] \mapsto \bigoplus_{[g] \in [G]} \left(\sum_{\tau \in \widehat{G}} \chi_\tau(g) [\text{Hom}^G(V_\tau, V)] \right).$$

Replacing X by $X \times S^1$, we obtain a similar description for an isomorphism

$$\Phi^1 : K_G^1(X)_\mathbb{C} \xrightarrow{\cong} K^1(X)_\mathbb{C} \otimes \text{Rep}(G)_\mathbb{C}.$$

In particular, we obtain surjective localization maps $\Phi_{[g]}^* : K_G^*(X)_\mathbb{C} \rightarrow K^*(X)_\mathbb{C}$ given by composing Φ^* with evaluation at $[g]$.

The following lemma allows a more transparent description of the localization maps $\Phi_{[g]}^*$ of Remark 5.10.

Lemma 5.11. *Suppose that the finite group G acts trivially on the compact space X and let V be a G -equivariant complex vector bundle over X giving a class $[V] \in K_G^0(X)_\mathbb{C}$. Fix $g \in G$, let $d := \text{ord}(g)$ denote the order of g , and let $C_d \subseteq \mathbb{T}$ the set of d th roots of unity. Then V decomposes into a direct sum of vector bundles $V = \bigoplus_{\zeta \in C_d} V_\zeta$ such that each V_ζ is the eigenbundle for the eigenvalue ζ for the action of g on V and we get*

$$\Phi_{[g]}^0([V]) = \sum_{\zeta \in C_d} \zeta [V_\zeta] \in K^0(X)_\mathbb{C}.$$

Replacing X by $X \times S^1$ (resp. the one-point compactification X^+), a similar description is obtained for $\Phi_{[g]}^1$ (resp. for the case of locally compact spaces X).

Proof. We have to prove that $\sum_{\zeta \in C_d} \zeta [V_\zeta] = \sum_{\tau \in \widehat{G}} \chi_\tau(g) [\text{Hom}^G(V_\tau, V)]$ in $K^0(X)_\mathbb{C}$. For this first observe that by the arguments of [47, Proposition 2.2] we get

$$V \cong \bigoplus_{\tau \in \widehat{G}} V_\tau \otimes \text{Hom}^G(V_\tau, V)$$

as G -equivariant vector bundles, where G acts trivially on the vector bundles $\text{Hom}^G(V_\tau, V)$. For each $\zeta \in C_d$ let $\chi_\zeta : \langle g \rangle \rightarrow \mathbb{T}$ denote the corresponding character of the cyclic group $\langle g \rangle$ generated by g , i.e., $\chi_\zeta(g^l) = \zeta^l$. Let V_τ^ζ denote the eigenspace in V_τ for the eigenvalue ζ of $\tau(g)$. Its dimension d_τ^ζ gives the multiplicity of the character χ_ζ in the restriction of τ to $\langle g \rangle$. Computing $\chi_\tau(g) = \text{trace } \tau(g)$ with respect to a corresponding basis, we get $\chi_\tau(g) = \sum_{\zeta \in C_d} d_\tau^\zeta \zeta$. We therefore get

$$(5.7) \quad \Phi_{[g]}^0([V]) = \sum_{\tau \in \widehat{G}} \chi_\tau(g) [\text{Hom}^G(V_\tau, V)] = \sum_{\zeta \in C_d} \sum_{\tau \in \widehat{G}} d_\tau^\zeta \zeta [\text{Hom}^G(V_\tau, V)].$$

On the other hand, using the isomorphism $V \cong \bigoplus_{\tau \in \widehat{G}} (V_\tau \otimes \text{Hom}^G(V, V_\tau))$ we get

$$V_\zeta \cong \bigoplus_{\tau \in \widehat{G}} (V_\tau^\zeta \otimes \text{Hom}^G(V_\tau, V)) \cong \bigoplus_{\tau \in \widehat{G}} \text{Hom}^G(V_\tau, V)^{d_\zeta^\tau},$$

so that we get the equality $[V_\zeta] = \sum_{\tau \in \widehat{G}} d_\zeta^\tau [\text{Hom}^G(V_\tau, V)]$ in $K^0(X)_\mathbb{C}$. Together with (5.7) this gives

$$\sum_{\zeta \in C_d} \zeta [V_\zeta] = \sum_{\zeta \in C_d} \sum_{\tau \in \widehat{G}} \zeta d_\zeta^\tau [\text{Hom}^G(V_\tau, V)] = \Phi_{[g]}^0([V]).$$

□

If a finite group acts on a compact space X , the module $K_G^*(X)_\mathbb{C}$ decomposes as

$$K_G^*(X)_\mathbb{C} \cong \bigoplus_{[g] \in [G]} K_G^*(X)_{\mathbb{C}, [g]}$$

and to classify the module it suffices to analyse the contributions $K_G^*(X)_{\mathbb{C}, [g]}$ from each conjugacy class $[g]$. The above lemma does this job in case where G acts trivially on X . In what follows next, we want to extend this description to the case of arbitrary compact G -spaces X .

Remark 5.12. Suppose that a finite group G acts on the compact space X . Then we always get a projection from $K_G^*(X)$ to the G -invariant part $K^*(X)^G$ of $K^*(X)$ which is given by sending the class $[V]$ of a G -equivariant vector bundle to the class $[V]$ with forgotten G -action. To see that V gives a G -invariant class in $K^0(X)$ observe that the action of an element $g \in G$ on V provides a continuous family of isomorphisms $\alpha_x^g : V_x \rightarrow V_{gx} = g^*(V)_x$, hence an isomorphism between the bundles V and g^*V . Note that the projection $K_G^*(X) \rightarrow K^*(X)^G$ becomes surjective after complexification, since for any G -invariant class $[V]$ the class $\frac{1}{|G|} [\bigoplus_{g \in G} g^*V] \in K_G^0(X)_\mathbb{C}$ maps to $[V]$ in the complexification of $K^0(X)^G$. Moreover, by Proposition 5.5 we see that the complexification of $K^*(X)^G$ is isomorphic to $K^*(G \backslash X)_\mathbb{C}$, so we obtain a canonical surjective projection

$$(5.8) \quad P_G^* : K_G^*(X)_\mathbb{C} \rightarrow K^*(G \backslash X)_\mathbb{C}$$

which sends a class $[V] \in K_G^0(X)$ to the class $\frac{1}{|G|} \sum_{g \in G} [g^*V] \in K^*(G \backslash X)_\mathbb{C}$. By passing to one-point compactifications, we obtain a similar projection for locally compact G -spaces.

Suppose now that X is a compact G -space for the finite group G . Let $g \in G$. Any G -equivariant vector bundle on X restricts to the set of g -fixed points $X^g = \{x \in X : gx = x\}$. It then decomposes into g -eigenbundles V_ζ as in Lemma 5.11. Let Z_g denote the centralizer of g in G . Each of the bundles V_ζ are Z_g -equivariant. Define a map

$$(5.9) \quad \phi_{[g]}^0 : K_G^0(X)_\mathbb{C} \longrightarrow K_{Z_g}^0(X^g)_\mathbb{C}, \quad \phi_{[g]}([V]) := \sum_{\zeta} \zeta [V_\zeta].$$

Composing this with the projections $P_{Z_g}^0 : K_{Z_g}^0(X^g)_\mathbb{C} \rightarrow K^0(Z_g \backslash X^g)_\mathbb{C}$ and summing over $[g]$ yields a map $\phi^0 : K_G^0(X)_\mathbb{C} \rightarrow \bigoplus_{[g] \in [G]} K^0(Z_g \backslash X^g)_\mathbb{C}$.

Replacing X by $X \times S^1$ (or by the one-point compactification X^+) and repeating the construction gives a map

$$(5.10) \quad \phi^* : K_G^*(X)_\mathbb{C} \rightarrow \bigoplus_{[g] \in [G]} K^*(Z_g \backslash X^g)_\mathbb{C}.$$

for any locally compact G -space X .

Consider now the group stabilizer bundle $\text{Stab}(X) = \{(x, g) : x \in X, g \in G_x\}$. G acts on $\text{Stab}(X)$ via $h(x, g) = (h \cdot x, hgh^{-1})$ and $\text{Stab}(X)$ decomposes as a disjoint union

$$\text{Stab}(X) \cong \bigsqcup_{[g] \in [G]} \bigsqcup_{[h] \in G/Z_g} h \cdot X^g \cong \bigsqcup_{[g] \in [G]} (G \times_{Z_g} X^g),$$

which implies a decomposition

$$(5.11) \quad G \backslash \text{Stab}(X) \cong \bigsqcup_{[g] \in [G]} Z^g \backslash X^g.$$

From this we obtain a decomposition

$$K^*(G \backslash \text{Stab}(X))_{\mathbb{C}} \cong \bigoplus_{[g] \in [G]} K^*(Z^g \backslash X^g)_{\mathbb{C}}.$$

Therefore we may regard the map (5.10) as having target $K^*(G \backslash \text{Stab}(X))_{\mathbb{C}}$ and we obtain a well defined map

$$(5.12) \quad \Phi_X : K_G^*(X)_{\mathbb{C}} \longrightarrow K^*(G \backslash \text{Stab}(X))_{\mathbb{C}} \cong \bigoplus_{[g] \in [G]} K^*(Z^g \backslash X^g)_{\mathbb{C}}.$$

Baum and Connes [7] prove the following theorem.

Theorem 5.13 (Baum-Connes). *If G is a finite group, X a G -space, then (5.12) is an isomorphism*

$$K_G^*(X)_{\mathbb{C}} \xrightarrow{\cong} K^*(G \backslash \text{Stab}(X))_{\mathbb{C}}$$

which sends the component $K_G^*(X)_{\mathbb{C}, [g]}$ of $K_G^*(X)_{\mathbb{C}}$ to the component $K^*(Z^g \backslash X^g)_{\mathbb{C}}$ of $K^*(G \backslash \text{Stab}(X))_{\mathbb{C}}$.

The idea of the proof is given by observing that for any induced G -space $X = G \times_H Y$ the assertion of the theorem is true if (and only if) it is true for the H -space Y . Together with the fact that for trivial G -spaces the theorem has been checked already in Lemma 5.11, this allows to combine induction over the order of G with a Meyer-Vietoris argument to obtain the proof.

We first want to verify that the map (5.12) is a natural transformation between two (compactly supported) cohomology theories on G -spaces, namely $X \mapsto K_G^*(X)_{\mathbb{C}}$ and $X \mapsto K^*(G \backslash \text{Stab}(X))_{\mathbb{C}}$. Indeed, let $Z \subseteq X$ be a closed, G -invariant subspace of a G -space X . Then $\text{Stab}(Z) \subset \text{Stab}(X)$ equivariantly so $G \backslash \text{Stab}(Z) \subset G \backslash \text{Stab}(X)$ naturally. Hence a G -invariant closed subspace of X yields a 6-term exact sequence

$$\cdots \xrightarrow{\partial} K^*(G \backslash \text{Stab}(X \setminus Z)) \longrightarrow K^*(G \backslash \text{Stab}(X)) \longrightarrow K^*(G \backslash \text{Stab}(Z)) \xrightarrow{\partial} \cdots$$

We leave the proof of the following lemma to the reader.

Lemma 5.14. *If $Z \subseteq X$ is a closed, G -invariant subspace, then the diagram (5.13)*

$$\begin{array}{ccccccc} \cdots & \longrightarrow & K_G^*(X \setminus Z)_{\mathbb{C}} & \longrightarrow & K_G^*(X)_{\mathbb{C}} & \longrightarrow & K_G^*(Z)_{\mathbb{C}} \longrightarrow \cdots \\ & & \downarrow \Phi_{X \setminus Z} & & \downarrow \Phi_X & & \downarrow \Phi_Z \\ \cdots & \longrightarrow & K^*(G \backslash \text{Stab}(X \setminus Z))_{\mathbb{C}} & \longrightarrow & K^*(G \backslash \text{Stab}(X))_{\mathbb{C}} & \longrightarrow & K^*(G \backslash \text{Stab}(Z))_{\mathbb{C}} \longrightarrow \cdots \end{array}$$

commutes.

Remark 5.15. If $X = G \times_H Y$ for some H -space Y , we obtain an isomorphism

$$R : K_G^*(X) \rightarrow K_H^*(Y)$$

given by composing the obvious restriction map $K_G^*(X) \rightarrow K_H^*(X)$ with the map $K_H^*(X) \rightarrow K_H^*(Y)$ coming from the inclusion of the open subset Y in X (see [47,

p. 132]). Its inverse $\text{Ind} : K_H^*(Y) \rightarrow K_H^*(X)$ is given on the level of vector bundles by $[V] \mapsto [G \times_H V]$. It is then easy to check that the diagram

$$(5.14) \quad \begin{array}{ccc} K_G^*(X) \otimes \text{Rep}(G) & \xrightarrow{m_G} & K_G^*(X) \\ R \otimes \text{res} \downarrow & & \downarrow R \\ K_H^*(Y) \otimes \text{Rep}(H) & \xrightarrow{m_H} & K_H^*(Y) \end{array}$$

commutes, where m_G and m_H denote the respective module actions of $\text{Rep}(G)$ and $\text{Rep}(H)$ and $\text{res} : \text{Rep}(G) \rightarrow \text{Rep}(H)$ is the restriction map.

Lemma 5.16. *Let G be a finite group, $H \subseteq G$ a subgroup and $X := G \times_H Y$ be an induced G -space. Then the result of Baum and Connes for H acting on Y implies the result of Baum and Connes for G acting on X .*

Proof. Recall that $G \times_H Y$ is the orbit space of the space $G \times Y$ for the (right) action $(\gamma, y)h = (\gamma h, h^{-1}y)$, of the subgroup H with G -action given by $g[\gamma, y] = [g\gamma, y]$.

Fix $g \in H$ and observe that

$$X^g = \{[\gamma, y] \in G \times_H Y \mid \gamma^{-1}g\gamma \in H_y\}.$$

We therefore obtain a map

$$\Phi' : X^g \rightarrow H \backslash \text{Stab}(Y); [\gamma, y] \mapsto [\gamma^{-1}g\gamma, y].$$

which is well defined, because if we replace (γ, y) by the pair $(\gamma h, h^{-1}y)$ for some $h \in H$, then $(\gamma^{-1}g\gamma, y)$ is replaced by $(h\gamma g\gamma^{-1}h, h^{-1}y) = (\gamma g\gamma^{-1}, y)h$. Moreover, if $z \in Z^g$, the centralizer of g in G , then one checks that $\Phi'(z[\gamma, y]) = \Phi'([\gamma, y])$, so that Φ' descends to a map

$$\Phi : Z^g \backslash X^g \rightarrow H \backslash \text{Stab}(Y).$$

As discussed above (see (5.11)), we can fibre the quotient of the group stabilizer bundle $H \backslash \text{Stab}(Y)$ over the set of conjugacy classes in H ,

$$H \backslash \text{Stab}(Y) = \bigsqcup_{[h] \in [H]} (Z^h \cap H) \backslash Y^h$$

where Z^h is the centralizer in G of a chosen representative h of $[h]$.

In this picture, the range of Φ is exactly the union of the components of $H \backslash \text{Stab}(Y)$ which are labelled by $[h]$ with h conjugate in G to g , and Φ induces a homeomorphism between $Z^g \backslash X^g$ and $\bigsqcup_{[h] \in [H], [h] \subseteq [g]} (Z^h \cap H) \backslash Y^h$. This shows that Φ induces an isomorphism

$$(5.15) \quad \Phi^* : \bigoplus_{[h] \in [H], [h] \subseteq [g]} K^*((Z^h \cap H) \backslash Y^h) \xrightarrow{\cong} K^*(Z^g \backslash X^g).$$

This can be viewed, in a rather trivial way, as a $\text{Rep}(G)$ -module isomorphism, with $\text{Rep}(G)$ -module structure given (on each side) by evaluation of characters at $[g]$.

Now, if $g \in G$ such that the conjugacy class $[g]_G$ of g in G does not intersect with H , then it follows from the commutativity of diagram (5.14) that $K_G^*(X)_{\mathbb{C}, [g]} = \{0\}$, since the restriction of the characteristic function of $[g]_G$ to H vanishes. On the other hand, if g is not conjugate to any element in H , it follows that $X^g = \emptyset$, so that we also have $K^*(Z^g \backslash X^g)_{\mathbb{C}} = \{0\}$.

Using the above results together with the fact that the action of $\text{Rep}(G)_{\mathbb{C}}$ on $K_H^*(Y)_{\mathbb{C}}$ is given via the restriction $\text{res} : \text{Rep}(G)_{\mathbb{C}} \rightarrow \text{Rep}(H)_{\mathbb{C}}$, we now obtain a

diagram of $\text{Rep}(G)_{\mathbb{C}}$ -module maps

$$\begin{array}{ccc} K_H^*(Y)_{\mathbb{C}} & \xrightarrow{\Phi_Y} & \bigoplus_{[g] \in [G]} \left(\bigoplus_{[h] \in [H], [h] \subseteq [g]} K^*((Z^h \cap H) \setminus Y^h)_{\mathbb{C}} \right) \\ \text{Ind} \downarrow & & \downarrow \Phi \\ K_G^*(X)_{\mathbb{C}} & \xrightarrow{\Phi_X} & \bigoplus_{[g] \in [G]} K^*(Z^g \setminus X^g)_{\mathbb{C}} \end{array}$$

We leave it as an exercise to the reader to check that this diagram commutes (do this first in case where Y is compact, and then obtain the general case by passing from Y to Y^+). Assume that the theorem of Baum and Connes holds for H acting on Y , i.e., Φ_Y is an isomorphism. Since Φ and Ind are also isomorphisms, the same must then be true for Φ_X . \square

We are now ready for the

Proof of Theorem 5.13. We induct on the order of the group G . For trivial groups there is nothing to prove. Suppose the result is true for all groups G of cardinality $\leq n$. Let X be a G -space, G finite, of cardinality $\leq n+1$. Let $F \subset X$ be the stationary set. From Lemma 5.14 the diagram

(5.16)

$$\begin{array}{ccccccc} \cdots & \longrightarrow & K_G^*(X \setminus F)_{\mathbb{C}} & \longrightarrow & K_G^*(X)_{\mathbb{C}} & \longrightarrow & K_G^*(F)_{\mathbb{C}} \longrightarrow \cdots \\ & & \downarrow \Phi_{X \setminus F} & & \downarrow \Phi_X & & \downarrow \Phi_F \\ \cdots & \longrightarrow & K^*(\text{Stab}(X \setminus F))_{\mathbb{C}} & \longrightarrow & K^*(\text{Stab}(X))_{\mathbb{C}} & \longrightarrow & K^*(\text{Stab}(F))_{\mathbb{C}} \longrightarrow \cdots \end{array}$$

commutes. The vertical map Φ_F is an isomorphism by Lemma 5.11. Thus, by the Five Lemma, it suffices to show that the first vertical map $\Phi_{X \setminus F}: K_G^*(X \setminus F) \rightarrow K^*(\text{Stab}(X \setminus F))$ is an isomorphism, too.

So from now on we may assume without loss of generality, that X does not contain any G -fixed points. In this case we find a cover $\{U_i : i \in I\}$ of G -invariant open subsets U_i of X such that each of these sets is G -homeomorphic to $G \times_{H_i} Y_i$ for some proper subgroup H_i of G . It follows then from the induction hypothesis together with Lemma 5.16 that the theorem holds for each U_i . By Meyer-Vietoris, using Lemma 5.14, this implies the theorem for all unions $U_F := \bigcup_{i \in F} U_i$, with $F \subseteq I$ finite. The result now follows from continuity of K -theory with respect to inductive limits and the fact that $C_0(X) \rtimes G = \lim_F C_0(U_F) \rtimes G$ and $C_0(G \setminus \text{Stab}(X)) = \lim_F C_0(G \setminus \text{Stab}(U_F))$. \square

Example 5.17. Consider the action of the dihedral group $G = D_4$ on \mathbb{T}^2 as introduced in Example 2.6. Recall that it is generated by the matrices $R, S \in \text{GL}(2, \mathbb{Z})$ with $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and the canonical action of $\text{GL}(2, \mathbb{Z})$ on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. This group of order eight has five conjugacy classes given by

$$[E], [R^2], [R], [S], [SR].$$

The corresponding centralizers are

$$\begin{aligned} Z^E &= Z^{R^2} = G, & Z^R &= \langle R \rangle, & Z^S &= \{E, R^2, S, SR^2\}, \\ Z^{RS} &= \{E, R^2, RS, SR\}, \end{aligned}$$

while the corresponding fixed-point sets are

$$\begin{aligned} X^E &= \mathbb{T}^2, & X^{R^2} &= \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}, & X^R &= \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\}, \\ X^S &= \{(z, 1) : z \in \mathbb{T}\} \cup \{(z, -1) : z \in \mathbb{T}\}, & X^{RS} &= \{(z, z) : z \in \mathbb{T}\}. \end{aligned}$$

With $Z = \{(s, t) : 0 \leq s, t \leq \frac{1}{2}, 0 \leq t \leq s\}$ as in Example 2.6 and $I = [0, 1]$ we get

$$Z^E \setminus X^E = G \setminus \mathbb{T}^2 \cong Z, \quad Z^S \setminus X^S \cong I \sqcup I, \quad Z^{RS} \setminus X^{RS} \cong I,$$

the set $Z^{R^2} \setminus X^{R^2}$ has three elements and $Z^R \setminus X^R = X^R$ has two elements. Thus, as a consequence of the theorem of Baum and Connes we see that

$$K_G^0(\mathbb{T}^2)_{\mathbb{C}} = \mathbb{C}^9 \quad \text{and} \quad K_G^1(\mathbb{T}^2)_{\mathbb{C}} = \{0\}.$$

The formula of Atiyah and Segal given in Theorem 5.9 follows from a little manipulation similar to that of the proof of Lemma 5.6, and Theorem 5.13. This is not hard to check, and the original reference explains this quite clearly, so we will omit the proof.

The result of Baum and Connes is of course stronger; it gives the dimensions of $K_G^0(X)_c$ and $K_G^1(X)_c$ separately.

5.2. Computation of integral K-theory. We now return to integral equivariant K-theory.

In the previous section, we explained the result of Baum and Connes, which gave a formula for the *ranks* of $K_G^i(X)$, $i = 0, 1$. A finitely generated abelian group is determined by its free part and torsion part. In this section we will discuss the torsion part of $K_G^*(X)$. Computing this – even for finite group actions – seems to be a much more difficult problem than computing the free part.

Since part of what we are going to describe is quite general and works for locally compact groups, we now let G be an arbitrary locally compact group and X a proper and G -compact G -space.

Let I_X be the ideal in $C_0(X) \rtimes G$ as in (3.7). Let $Q_X := (C_0(X) \rtimes G)/I_X$. Then the Morita equivalence $C_0(G \setminus X) \cong I_X$ and excision yields a six-term exact sequence

$$(5.17) \quad \begin{array}{ccccc} K^0(G \setminus X) & \longrightarrow & K_G^0(X) & \longrightarrow & K_0(Q_X) \\ \partial \uparrow & & & & \downarrow \partial \\ K_1(Q_X) & \longleftarrow & K_G^1(X) & \longleftarrow & K^1(G \setminus X). \end{array}$$

In the case of group actions in which fixed-points are isolated (more precisely below), this exact sequence becomes quite tractable and we will study this situation in more detail below. In effect, this is a situation in which the Fell topology on $G \setminus \text{Stab}(X)^\wedge$ is very easy to understand.

When fixed-point sets are not just zero-dimensional, the problem obviously becomes more complicated. We are not going to consider the general case here. However, we will analyse Example 5.24 in some detail.

If the set of points in $G \setminus X$ with non-trivial isotropy is a discrete subset, then by Lemma 3.9, the C^* -algebra Q_X is isomorphic to a direct sum of compact operators, with one summand contributed by each pair (Gx, σ) , where Gx is an orbit with nontrivial stabilizer subgroup G_x , and $\sigma \neq 1_{G_x}$ in \hat{G}_x is an irreducible representation different from the trivial representation.

More formally, if \mathcal{I} denotes the set of orbits with nontrivial stabilizers,

$$(5.18) \quad Q_X \cong \bigoplus_{Gx \in \mathcal{I}} \left(\bigoplus_{\sigma \in \hat{G}_x \setminus \{1_{G_x}\}} \mathcal{K}(\mathcal{H}_{U\sigma}) \right),$$

where for each orbit $Gx \in \mathcal{I}$ we choose one representative x of that orbit and where $\mathcal{H}_{U\sigma}$ is as in (3.1). If we write $\text{Rep}^*(G_x)$ for the subgroup of $\text{Rep}(G)$ generated by the non-trivial irreducible representations of G_x it follows that $K_0(Q_X) \cong$

$\bigoplus_{Gx \in \mathcal{I}} \text{Rep}^*(G_x)$ is a free abelian group and $K_1(Q_X) = \{0\}$, so that the six-term sequence (5.17) becomes

$$(5.19) \quad 0 \longrightarrow K^0(G \backslash X) \longrightarrow K_G^0(X) \longrightarrow \bigoplus_{Gx \in \mathcal{I}} \text{Rep}^*(G_x) \xrightarrow{\partial} K^1(G \backslash X) \longrightarrow K_G^1(X) \longrightarrow 0.$$

In particular we see that $K^0(G \backslash X)$ injects into $K_G^*(X)$ and $K^1(G \backslash X)$ surjects to $K_G^1(X)$, in the case of isolated fixed-points. Note that, in case of finite G , according to Theorem 5.13 of Baum and Connes; $K^i(G \backslash X)$ injects in both after complexification: it is the contribution of the trivial conjugacy class [1]. Therefore, our statement adds to theirs that this is an injection in dimension zero *integrally*, and that $K^1(G \backslash X) \rightarrow K_G^1(X)$ is a surjection, likewise, even integrally.

As we will see, non-vanishing of $\partial([\sigma])$ for a generator in $K_0(Q_X)$ corresponding to some non-trivial representation σ of one of the isotropy groups G_x , obstructs extending the corresponding induced G -equivariant vector bundle $[V_\sigma]$ on the orbit of x to a G -equivariant vector bundle on X . The statement is that these obstructions are torsion: they vanish after multiplication by a suitable integer. Thus nV_σ can always be extended, for appropriate n .

We also note that the point of view suggested by (5.19) yields a somewhat different formula for the Euler characteristic. For a finite group G , let \widehat{G}^* denote $\widehat{G} - \{1\}$ where 1 is the trivial representation.

Proposition 5.18. *Let G be a discrete group acting properly and co-compactly on X such that the set D of points $x \in X$ with non-trivial isotropy is discrete. Then*

$$\text{Eul}(G \ltimes X) = \text{Eul}(G \backslash X) + \sum_{Gx \in G \backslash D} |\widehat{G}_x^*|.$$

We point this out mainly to emphasise that we are counting in a ‘transverse’ direction to Atiyah, Baum and Connes, where our theorems intersect, when G itself is a finite group. They fibre $G \backslash \text{Stab}(X)$ over the conjugacy classes, while we are fibring it over $G \backslash X$. To check that the two formulas for the Euler characteristic are the same, it suffices to observe that both – clearly – agree with the Euler characteristic in ordinary K-theory of the same Hausdorff space $G \backslash \text{Stab}(X)$.

In any case, since both the Baum-Connes formula and ours contain the term $\text{Eul}(G \backslash X)$, we can remove it from each side. The identity

$$(5.20) \quad \sum_{[g] \in [G] \setminus \{[1]\}} \text{card}(Z^g \backslash X^g) = \sum_{Gx \in I} |\widehat{G}_x^*|.$$

is a good exercise to prove directly (the two sides of it are, roughly speaking, in a relation of duality, since if X is a point, the left-hand-side computes the number of nontrivial conjugacy classes in G and the right-hand-side computes the number of nontrivial irreducible representations.)

To compute equivariant K-theory in the case of isolated fixed points, it remains to solve the following

Problem: Describe the boundary map

$$\partial: \bigoplus_{Gx \in \mathcal{I}} \bigoplus_{\sigma \in \widehat{G}_x \setminus \{1_{G_x}\}} \mathbb{Z} \longrightarrow K^1(G \backslash X)$$

in (5.19).

Remark 5.19. In the case of a *free* action there is of course nothing to do. In this case the quotient term vanishes because Q_X is itself the zero C^* -algebra.

If (c.f. Example 5.7) there is a *single* point x_0 in $G \backslash X$ with non-trivial isotropy, and if this point is fixed by the entire group G (so that G must be compact) then (5.19) becomes

$$(5.21) \quad 0 \longrightarrow K^0(G \backslash X) \longrightarrow K_G^0(X) \longrightarrow \text{Rep}^*(G) \xrightarrow{\partial} K^1(G \backslash X) \longrightarrow K_G^1(X) \longrightarrow 0.$$

The quotient map $K_G^0(X) \rightarrow \text{Rep}^*(G)$ in this sequence is induced by the G -map $\text{pt} \rightarrow X$ corresponding to the stationary point. If X is compact, this map is split by the map $X \rightarrow \text{pt}$ and hence the boundary map ∂ vanishes in this case. Hence

$$K_G^0(X) \cong K^0(G \backslash X) \bigoplus \text{Rep}^*(G), \quad K_G^1(X) \cong K^1(G \backslash X).$$

The general case is as follows.

Theorem 5.20. *Suppose that the locally compact group G acts properly on X such that the following are satisfied:*

- (i) (G, X) satisfies Palais's slice property (SP);
- (ii) the orbits Gx with nontrivial stabilizers G_x are isolated in $G \backslash X$; and
- (iii) all stabilizers are finite.

Then the boundary map $\partial : K_0(Q_X) \rightarrow K_G^1(X)$ of (5.19) is rationally trivial (i.e., the image of ∂ is a torsion subgroup of $K^1(G \backslash X)$).

Moreover, if an element $\nu \in K_0(Q_X)$ is an element of the summand $\bigoplus_{\sigma \in \widehat{G}_x \setminus \{1_{G_x}\}} K_0(\mathcal{K}(\mathcal{H}_{U^\sigma}))$ in the direct-sum decomposition (5.18) of $K_0(Q_X)$, then the order of $\partial(\nu)$ is a divisor of the order $|G_x|$.

The following is a well-known fact about flat vector bundles.

Lemma 5.21. *Let H be a finite group acting freely on the compact space Y . Then for each $\sigma \in \widehat{H}$, the difference of classes*

$$[Y \times V_\sigma] - \dim(V_\sigma)[1_Y] \in K_H^0(Y)$$

is torsion of order a divisor of $|H|$. Thus,

$$0 = |H| \cdot (\dim(V_\sigma) \cdot [Y \times \mathbb{C}] - [Y \times V_\sigma]) \in K_H^0(Y).$$

Proof. If H acts freely on Y we have $K_H^0(Y) \cong K^0(H \backslash Y)$ via sending the class of an equivariant vector bundle V over Y to the class of the bundle $H \backslash V$ over $H \backslash Y$ (e.g. see [47, Proposition 2.1]). Consider the sequence

$$K^0(H \backslash Y) \xrightarrow{q^*} K^0(Y) \xrightarrow{q_*} K^0(H \backslash Y)$$

in which the first map is given by the pull-back of vector bundles with respect to the quotient map $q : Y \rightarrow H \backslash Y$ and the second map is the transfer map which sends a vector bundle V over Y to the bundle with fibre $\bigoplus_{g \in H} V_{gy}$ over $Hy \in H \backslash Y$. It is clear that the composition $q_* \circ q^*$ acts on $K^0(H \backslash Y)$ as multiplication by $|H|$.

If $\sigma \in \widehat{H}$, then the H -bundle $[Y \times V_\sigma] \in K_H^0(Y)$ (with respect to the diagonal action) corresponds to the (flat) bundle $[Y \times_H V_\sigma] \in K^0(H \backslash Y)$. The pull-back $q^*([Y \times_H V_\sigma]) \in K^0(Y)$ is the class of the trivial bundle $[Y \times V_\sigma] = \dim(V_\sigma)[Y \times \mathbb{C}]$ which is mapped to $|H| \dim(V_\sigma)[(H \backslash Y) \times \mathbb{C}]$ by the transfer map $q_* : K^0(Y) \rightarrow K^0(H \backslash Y)$. Thus we see that

$$|H|[Y \times_H V_\sigma] = q_* \circ q^*([Y \times_H V_\sigma]) = |H| \dim(V_\sigma)[(H \backslash Y) \times \mathbb{C}]$$

and the result follows. \square

In what follows next we want to consider the case of an induced space $X = G \times_H Y$, where H is a finite subgroup of the locally compact group G and Y is a compact H -space with isolated H -fixed point $y \in Y$ such that H acts freely on $Y \setminus \{y\}$. Regarding Y as a closed subspace of X , it is clear that Gy is the only non-free orbit in X , and its stabilizer is $G_y = H$, so that the K-theory of $Q_X = (C_0(X) \rtimes G)/I_X$ is the free abelian group $\text{Rep}^*(H)$ generated by the representations $\sigma \in \widehat{H} \setminus \{1_H\}$.

Note also that if X is a G -space and Z is a closed G -invariant subspace of X , then the restriction map $\text{res}_Z : C_0(X) \rightarrow C_0(Z)$ induces a quotient map

$$\text{res}_Z \rtimes G : C_0(X) \rtimes G \rightarrow C_0(Z) \rtimes G.$$

Proposition 5.22. *Let $X = G \times_H Y$ and $Q_X = (C_0(X) \rtimes G)/I_X$ be as above. Let $\nu_\sigma \in K_0(Q_X)$ be the class corresponding to the given representation $\sigma \in \widehat{H} \setminus \{1_H\}$. Then there exists a class $\mu_\sigma \in K_G^0(X)$ such that the following are true*

- (i) *If $q_X : C_0(X) \rtimes G \rightarrow Q_X$ denotes the quotient map, then $q_{X,*}(\mu_\sigma) = |H|\nu_\sigma$.*
- (ii) *if Z is any closed G -invariant subspace of X which does not contain the orbit Gy , then $(\text{res}_Z \rtimes G)_*(\mu_\sigma) = 0$ in $K_0(C_0(Z) \rtimes G)$.*

Proof. If we regard Y as a closed subspace of $X = G \times_H Y$ via $y \mapsto [e, y]$, we see that $Z \cong G \times_H Y_Z$ with $Y_Z := Y \cap Z$. It is not difficult to check that the canonical Morita equivalence $C_0(G \times_H Y) \rtimes G \sim_M C(Y) \rtimes H$, as explained before Proposition 3.13, restricts to the canonical Morita equivalence $C_0(G \times_H Y_Z) \rtimes G \sim_M C(Y_Z) \rtimes H$ and it follows from Corollary 3.14 and the Rieffel correspondence (see the discussion before Corollary 3.14) that this factors through a Morita equivalence $Q_X \sim_M Q_Y$. Thus these Morita equivalences provide us with a commutative diagram

$$\begin{array}{ccccc} K_0(C(Y_Z) \rtimes H) & \xleftarrow{(\text{res}_{Y_Z} \rtimes H)_*} & K_0(C(Y) \rtimes H) & \xrightarrow{q_{Y,*}} & K_0(Q_Y) \\ \cong \downarrow & & \cong \downarrow & & \downarrow \cong \\ K_0(C_0(Z) \rtimes G) & \xleftarrow{(\text{res}_Z \rtimes G)_*} & K_0(C(X) \rtimes G) & \xrightarrow{q_{X,*}} & K_0(Q_X). \end{array}$$

Thus we may assume that $G = H$ and $Z = Y_Z$. This also allows the use of vector bundles for the description of K-theory classes. In this picture, the quotient map $q_{Y,*} : K_0(C(Y) \rtimes H) \rightarrow K_0(Q_Y)$ translates to the map $\tilde{q} : K_H^0(Y) \rightarrow K_0(Q_Y)$ which maps an H -bundle V over Y to the class $\sum_{\tau \neq 1_H} n_\tau \cdot \nu_\tau \in K_0(Q_Y)$, where the sum is over the non-trivial irreducible representations of H and n_τ denotes the multiplicity of τ in the representation of H on the fibre V_y . Define

$$\mu_\sigma := |H| \cdot ([Y \times V_\sigma] - \dim(V_\sigma) \cdot [Y \times \mathbb{C}]) \in K_H^0(Y) \cong K_G^0(X).$$

Then $\tilde{q}(\mu_\sigma) = |H| \cdot \nu_\sigma$ and $(\text{res}_{Y_Z} \rtimes F)_*(\mu_\sigma) = |H| \cdot ([Y_Z \times V_\sigma] - \dim(V_\sigma) \cdot [Y_Z \times \mathbb{C}]) = 0$ by Lemma 5.21 (because H acts freely on $Y \setminus \{y\}$.) \square

We are now ready for the

Proof of Theorem 5.20. For the proof it suffices to show that the canonical generators of $K_0(Q_X)$ are mapped to elements of finite order in $K_1(C_0(G \setminus X))$ under the boundary map $\partial : K_0(Q_X) \rightarrow K_1(C_0(G \setminus X))$. If the action of G on X is free, then $Q_X = \{0\}$ and the result is trivial. So assume now that Gy is any fixed orbit with non-trivial stabilizer G_y and let ν_σ denote the generator of $K_0(Q_X)$ corresponding to the representation $\sigma \in \widehat{G_y} \setminus \{1_{G_y}\}$. We claim that $|G_y|\partial(\nu_\sigma) = 0$. This will also prove the last statement of the theorem.

Since, by assumption, there exists a neighborhood of Gy in $G \setminus X$ such that all orbits in this neighborhood are free, and since the actions of G on X satisfies the slice property (SP), we may choose an open G -invariant neighborhood U of y with G -compact closure $W = \bar{U}$ with the following properties:

- (i) $W \cong G \times_{G_y} Y$ for some compact G_y -space Y .
- (ii) G acts freely on $W \setminus \{Gy\}$.

Let $Z = W \setminus U$. Then Proposition 5.22 implies that we can find a class $\mu_\sigma \in K_0(C_0(W) \rtimes G)$ such that $q_{W,*}(\mu_\sigma) = |G_y| \cdot \nu_\sigma$ and $(\text{res}_Z \rtimes G)_*(\mu_\sigma) = 0$. Applying the Meyer-Vietoris sequence in K-theory (e.g. see [9, Theorem 21.5.1]) to the pull-back diagram

$$\begin{array}{ccc} C_0(X) \rtimes G & \xrightarrow{\text{res}_{X \setminus U} \rtimes G} & C_0(X \setminus U) \rtimes G \\ \text{res}_W \rtimes G \downarrow & & \downarrow \text{res}_Z \rtimes G \\ C_0(W) \rtimes G & \xrightarrow{\text{res}_Z \rtimes G} & C_0(Z) \rtimes G \end{array}$$

we see that we may glue the class $\mu_\sigma \in K_0(C_0(W) \rtimes G)$ with the zero-class in $K_0(C_0(X \setminus U) \rtimes G)$ to obtain a class $\mu \in K_0(C_0(X) \rtimes G)$ such that $q_{X,*}(\mu) = |G_y| \cdot \nu_\sigma \in K_0(Q_X)$. This implies that $\partial(|G_y| \cdot \nu_\sigma) = 0$ in $K^1(G \setminus X)$. \square

In the above argument, we showed that $\text{Rep}^*(H)$ is a direct summand of $K_0(Q_X)$ in the case of isolated fixed points with finite stabilizer H , and we computed that the composition

$$\text{Rep}^*(H) \subseteq \text{Rep}(H) \cong K_G^0(Gy) \rightarrow K_0(Q_X) \xrightarrow{\partial} K^1(G \setminus X)$$

maps any generator $[\sigma] \in \text{Rep}^*(H)$ to a torsion class in $K^1(G \setminus X)$.

If X is a smooth manifold and the Lie group G acts smoothly, this composition may be made slightly more explicit using the language of differential topology. In the notation of the proof of Theorem 5.20, take a point $y \in X$ with isotropy H . Let ν be the normal bundle to Gy ; we may equip it with a G -invariant Riemannian metric, then a re-scaling composed with the exponential map determines a tubular neighbourhood embedding $\nu \cong U \subset X$ for an open G -invariant neighbourhood U of Gy . Thus, $U \cong G \times_H Y$ for the corresponding orthogonal linear action of H on a Euclidean space $Y := \nu_y \cong \mathbb{R}^k$. Let $S\nu$ be the unit sphere bundle of ν , let $i: S\nu \rightarrow U \setminus Gy$ be the corresponding smooth equivariant embedding; its normal bundle is equivariantly trivializable (it is isomorphic to a trivial G -vector bundle with trivial G -action). Thus, there is a G -equivariant smooth open embedding $S\nu \times \mathbb{R} \rightarrow U \setminus Gy$. If fixed-points are isolated, then G acts freely on $S\nu$ and $U \setminus Gy$ so that we obtain an open embedding

$$\varphi: G \setminus S\nu \times \mathbb{R} \longrightarrow G \setminus X.$$

Now the boundary map $\partial: \text{Rep}^*(H) \rightarrow K^1(G \setminus X)$ may be described simply as follows: it is the composition

$$(5.22) \quad \text{Rep}^*(H) \subset \text{Rep}(H) \cong K_G^0(Gy) \xrightarrow{p^*} K_G^0(S\nu) \cong K_G^0(G \setminus S\nu) \xrightarrow{\otimes \beta} K^1(G \setminus S\nu \times \mathbb{R}) \xrightarrow{\varphi!} K^1(G \setminus X),$$

where $p: S\nu \rightarrow Gy$ is the projection map, $\beta \in K^1(\mathbb{R})$ the Bott class.

This construction, as mentioned above, produces torsion classes in $K^1(G \setminus X)$ of order a divisor of $|H|$. We will see in Example 5.26 below that these classes are not always trivial, so that the boundary map in (5.19) does not always vanish in general.

But we first present an interesting example of an action with isolated orbits with non-trivial stabilizers, in which $K^1(G \setminus X) = \{0\}$, so that the exact sequence (5.19) computes everything.

Example 5.23. In this example we consider the cyclic group $H = \langle R \rangle$ of order four with $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \text{GL}(2, \mathbb{Z})$ acting on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. This is the action

of the dihedral group $G = \langle R, S \rangle$ on \mathbb{T}^2 as considered in Examples 2.6, 3.5 and 4.10 restricted to the subgroup $H \subseteq G$. Deformations of the crossed product $C(\mathbb{T}^2) \rtimes H$, known as non-commutative 2-spheres, have been studied extensively in the literature, and it is shown in [18] that the K-theory groups of these deformations are isomorphic to the K-theory groups of $C(\mathbb{T}^2) \rtimes H$.

If we study this action on the fundamental domain $\{(s, t) : -\frac{1}{2} \leq s, t \leq \frac{1}{2}\} \subseteq \mathbb{R}^2$ for the action of \mathbb{Z}^2 on \mathbb{R}^2 , we see that (the image in \mathbb{T}^2 of) $\{(s, t) : 0 \leq s, t \leq \frac{1}{2}\}$ in \mathbb{T}^2 is a fundamental domain (but not a topological fundamental domain as in Definition 2.4) for the action of H on \mathbb{T}^2 such that in the quotient $H \backslash \mathbb{T}^2$ the line $\{(s, 0) : 0 \leq s \leq \frac{1}{2}\}$ in the boundary is glued to $(0, t) : 0 \leq t \leq \frac{1}{2}$ and the line $\{(s, \frac{1}{2}) : 0 \leq s \leq \frac{1}{2}\}$ is glued to $(\frac{1}{2}, t) : 0 \leq t \leq \frac{1}{2}$. Thus we see that the quotient $H \backslash \mathbb{T}^2$ is homeomorphic to the 2-sphere S^2 .

There are only three orbits in $H \backslash \mathbb{T}^2$ with nontrivial stabilizers: the points corresponding to $(0, 0)$ and $(\frac{1}{2}, \frac{1}{2})$ in the fundamental domain have full stabilizer H and the orbit of the point corresponding to $(0, \frac{1}{2})$ has stabilizer $\langle R^2 \rangle$. Since $K^0(S^2) = \mathbb{Z}$, $K^1(S^2) = \{0\}$ and since H has three non-trivial characters and $\langle R^2 \rangle$ has one non-trivial character, we see instantly from the exact sequence (5.19), that $K_0((C(\mathbb{T}^2) \rtimes H) \cong \mathbb{Z}^8$ and $K_1(C(\mathbb{T}^2) \rtimes H) = \{0\}$.

In [18] we used a quite different and more complicated method for computing the K-theory groups of $C(\mathbb{T}^2) \rtimes H$. See [18] for an explicit description of the K-theory generators.

We proceed with an example of the equivariant K-theory computation for the action of $G = D_4$ on \mathbb{T}^2 as studied earlier in Examples 2.6, 3.5 and 4.10. In this case the orbits with non-trivial stabilizers are not isolated and we use the description of the ideal structure as given in Example 4.10 for the computation.

Example 5.24. Consider the crossed product $C(\mathbb{T}^2) \rtimes G$. It is shown in Example 4.10 that we get a sequence of ideals

$$\{0\} = I_0 \subseteq I_1 \subseteq I_2 \subseteq I_3 = C(\mathbb{T}^2) \rtimes G$$

with I_1 Morita equivalent to $C(Z)$, I_2/I_1 Morita equivalent to $C(\partial Z)$ and I_3/I_2 Morita equivalent to \mathbb{C}^8 .

We first compute the K-theory of I_2 . Since $K_0(I_1) = \mathbb{Z}$ and $K_1(I_1) = \{0\}$ and $K_0(I_2/I_1) = K_1(I_2/I_1) = \mathbb{Z}$, the six-term sequence with respect to the ideal $I_1 \sim_M C(Z)$ reads

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & K_0(I_2) & \longrightarrow & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ \mathbb{Z} & \longleftarrow & K_1(I_2) & \longleftarrow & 0 \end{array}$$

On the other hand, the ideal $J := C_0(\overset{\circ}{Z}, M^8(\mathbb{C})) \subseteq I_2$ with quotient I_2/J Morita equivalent to $C(\mathbb{T} \times \{0, 1\})$ gives a six-term sequence

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & K_0(I_2) & \longrightarrow & \mathbb{Z}^2 \\ \uparrow & & & & \downarrow \\ \mathbb{Z}^2 & \longleftarrow & K_1(I_2) & \longleftarrow & 0 \end{array}$$

It follows that $K_0(I_2) \cong \mathbb{Z}^2$ and $K_1(I_2) = \mathbb{Z}$. We then get the six-term sequence

$$\begin{array}{ccccc} \mathbb{Z}^2 & \longrightarrow & K_0(C(\mathbb{T}^2) \rtimes G) & \longrightarrow & \mathbb{Z}^8 \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & K_1(C(\mathbb{T}^2) \rtimes G) & \longleftarrow & \mathbb{Z} \end{array}$$

We leave it as an interesting exercise for the reader (using the structure of $I_3 = C(\mathbb{T}^2) \rtimes G$ as indicated in Example 3.5) to check directly that the class of $K_0(I_3/I_2)$ corresponding to the character $\chi_2 \in \widehat{G}$ at the point $(0, 0) \in Z$ cannot be lifted to a class in $K_0(I_3)$, and thus maps to a non-zero element in $K_1(I_2) \cong \mathbb{Z}$. Indeed, with a bit of work one can show that it maps to a generator of $K_1(I_2) \cong \mathbb{Z}$ which then implies that $K_1(C(\mathbb{T}^2) \rtimes G) = \{0\}$ and $K_0(C(\mathbb{T}^2) \rtimes G) = \mathbb{Z}^9$.

We should mention that the algebra $C(\mathbb{T}^2) \rtimes D_4$ of the above example is just the group algebra $C^*(\mathbb{Z}^2 \rtimes D_4)$ of the cristallographic group $\mathbb{Z}^2 \rtimes D_4$. The K-theory of this group algebra (together with the K-theories of all other crystallographic groups of rank 2) has been computed by completely different methods by Lück and Stamm in [38]. Even before that, the K-theory of the group algebras of crystallographic groups of rank 2 were computed by Yang in [50] by methods much closer to ours. In fact, many of the general results obtained in this article have been obtained in [50] in case of finite group actions.

We now provide the promised counter-example for the trivialization problem as posed in Remark 4.11.

Example 5.25. Let $G = \mathbb{Z}/n$ be a finite cyclic group acting linearly on some \mathbb{R}^m such that the action on $\mathbb{R}^m \setminus \{0\}$ is free (it follows that $n = 2$ or $m = 2k$ is even). Consider the crossed-product $C(B) \rtimes G$, where B denotes the closed unit ball in \mathbb{R}^n . Let $\text{res} : C(B) \rightarrow C(S^{m-1})$ denote the restriction map. We then obtain a map

$$(5.23) \quad \phi : R(G) \cong K_0(C(B) \rtimes G) \xrightarrow{\text{res}_*} K_0(C(S^{m-1}) \rtimes G) \cong K^0(G \backslash S^{m-1}).$$

It is shown in [2] (see also [26, Theorem 0.1]) that this map is surjective.

Using the bundle structure of $C(B) \rtimes G \cong C(B \times_G M_n(\mathbb{C}))$, this map can be described as follows: First of all consider $C(B \times_G M_n(\mathbb{C}))$ as a bundle over $[0, 1]$ with fibre $C^*(G) \cong M_n(\mathbb{C})^G$ at 0 and fibre $C(S^{m-1}) \rtimes G \cong C(S^{m-1} \times_G M_n(\mathbb{C}))$ at every $t \neq 0$. This bundle is trivial outside zero, and we obtain the above sequence of K-theory maps by first extending a projection of the zero fiber to a projection on a small neighborhood, then use triviality outside zero to extend the projection to the whole bundle, and finally evaluate this extended projection at the fiber at 1.

Assume now that $C(B \times_G M_n(\mathbb{C}))$ would be trivializable in the sense that it would be stably isomorphic to some subbundle of the trivial bundle $C(G \backslash B, \mathcal{K})$ with full fibres outside (the orbit of) the origin. Then any projection in the fiber at the origin has a trivial extension (by the constant section) to all of $C(G \backslash B, \mathcal{K})$ and the restriction of such projection to $G \backslash S^{m-1}$ would lie in the subgroup $\mathbb{Z} \cdot 1_{G \backslash S^{m-1}}$ of $K^0(G \backslash S^{m-1})$. Thus the existence of a trivialization together with surjectivity of the map res_* in (5.23) would imply that $K^0(G \backslash S^{m-1}) \cong \mathbb{Z} \cdot 1_{G \backslash S^{m-1}}$. But one can see in the appendix of [26] that $\widetilde{K}^0(G \backslash S^{m-1}) := K^0(G \backslash S^{m-1}) / \mathbb{Z} \cdot 1_{G \backslash S^{m-1}}$ is non-trivial (of finite order) for many choices of groups \mathbb{Z}_n acting on a suitable \mathbb{R}^m . For instance: if $m = 4$ and $n = 2$, we get $\widetilde{K}^0(G \backslash S^3) \cong \mathbb{Z}/2$.

The above example also provides the basis for the following example of an action with isolated fixed points, such that the boundary map in (5.19) does not vanish. We are grateful to Wolfgang Lück for pointing out this example to us.

Example 5.26. Choose n and m and an action of $G = \mathbb{Z}/n$ on the unit ball $B \subseteq \mathbb{R}^m$ as in the previous example such that $\widetilde{K}^0(G \backslash S^{m-1}) := K^0(G \backslash S^{m-1}) / \mathbb{Z} \cdot 1_{G \backslash S^{m-1}}$ is non-trivial. Notice that the map ϕ of (5.23) factors through a surjective map

$$\widetilde{\phi} : \text{Rep}^*(G) \rightarrow \widetilde{K}^0(G \backslash S^{m-1}),$$

since $\mathbb{Z} \cdot 1_{G \setminus S^{m-1}}$ is the image of $\mathbb{Z} \cdot 1_G \subseteq \text{Rep}(G)$. Glueing two such balls at the boundary $\partial B = S^{m-1}$ we obtain an action of G on S^m which fixes the two points $(0, \dots, 0, \pm 1)$ and which is free on $S^m \setminus \{(0, \dots, 0, \pm 1)\}$.

Let us consider $C(S^m) \rtimes G$ as a bundle over $[-1, 1]$ with fiber $C(S^{m-1}) \rtimes G$ over each $t \neq \pm 1$ and fiber $C^*(G)$ at $t = \pm 1$. We write $\text{Rep}(G)_+$ and $\text{Rep}(G)_-$ for the representation ring of G when identified with the K-theory of the fiber at $t = 1$ and $t = -1$, respectively. Then it follows from the description of the map ϕ in (5.23) of the previous example that for a given pair $([p_+], [p_-]) \in \text{Rep}(G)_+ \oplus \text{Rep}(G)_-$ there exists a class $[p] \in K_0(C(S^m) \rtimes G)$ restricting to the pair if and only if $\phi([p_+]) = \phi([p_-])$.

Now let I_{S^m} and Q_{S^m} as in Theorem 5.20. Then

$$K_0(Q_{S^m}) \cong \text{Rep}^*(G)_+ \bigoplus \text{Rep}^*(G)_-.$$

and the sequence (5.19) becomes

$$(5.24) \quad 0 \longrightarrow K^0(G \setminus S^m) \longrightarrow K_0(C(S^m) \rtimes G) \\ \longrightarrow \text{Rep}^*(G)_+ \bigoplus \text{Rep}^*(G)_- \xrightarrow{\partial} K^1(G \setminus S^m) \longrightarrow K_1(C(S^m) \rtimes G) \longrightarrow 0.$$

Since a pair $([p_+], [p_-]) \in \text{Rep}^*(G)_+ \oplus \text{Rep}^*(G)_-$ lies in the kernel of the boundary map ∂ if and only if it extends to a class in $K_0(C(S^m) \rtimes G)$, it follows from the above considerations that this is possible if and only if the values $\phi([p_+])$ and $\phi([p_-])$ differ by some class in $\mathbb{Z} \cdot 1_{G \setminus S^{m-1}}$. Thus we see that

$$\ker \partial = \{([p_+], [p_-]) \in \text{Rep}^*(G)_+ \bigoplus \text{Rep}^*(G)_- : \tilde{\phi}([p_+]) = \tilde{\phi}([p_-])\}$$

which is a proper subgroup of $\text{Rep}^*(G)_+ \bigoplus \text{Rep}^*(G)_-$. Thus it follows that the boundary map in (5.24) is not trivial.

Note that a more elaborate study of this action (which we omit) reveals that $K^1(G \setminus S^m) \cong \tilde{K}^0(G \setminus S^{m-1})$ and that the boundary map in (5.24) is given by sending a pair $([p_+], [p_-])$ to the difference $\tilde{\phi}([p_+]) - \tilde{\phi}([p_-])$ and hence is surjective. Thus it follows that

$$K_G^0(S^m) = K_0(C(S^m) \rtimes G) \cong K^0(G \setminus S^m) \bigoplus \ker \partial \quad \text{and} \quad K_G^1(S^m) = \{0\}.$$

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